EFFICIENT PREPARATION OF THE GIBBS STATE OF THE 2D TORIC CODE

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CRC TRR 352

Via Dissipation

Based on
arXiv:2510.03090
with


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(TU Munich)


Angelo Lucia
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Li Gao
(U Wuhan)


David Pérez-García
(UC Madrid)


Antonio Pérez-Hernández
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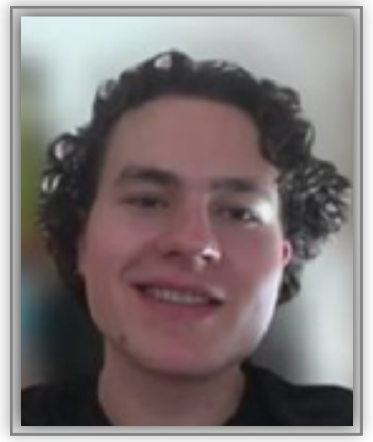
Cambyse Rouzé
(Inria Saclay)


Simone Warzel
(TU Munich)

Via Dualities

and
arXiv:2508.00126
with


Pablo Páez-Velasco
(UC Madrid)


Niclas Schilling
(U. Tübingen)


Samuel Scalet
(U. Cambridge)

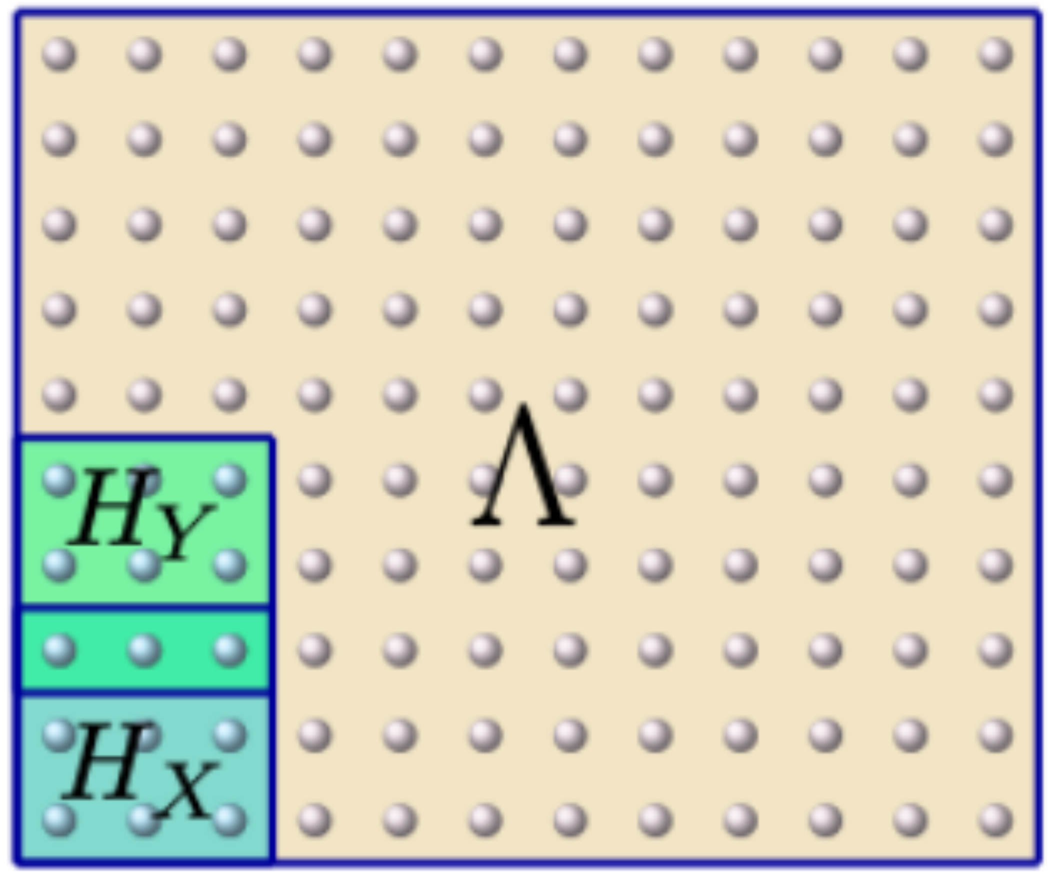

Frank Verstraete
(U. Cambridge)

SETTING: QUANTUM MANY-BODY SYSTEMS

- Spin lattice: $\Lambda \subset\subset \mathbb{Z}^D$

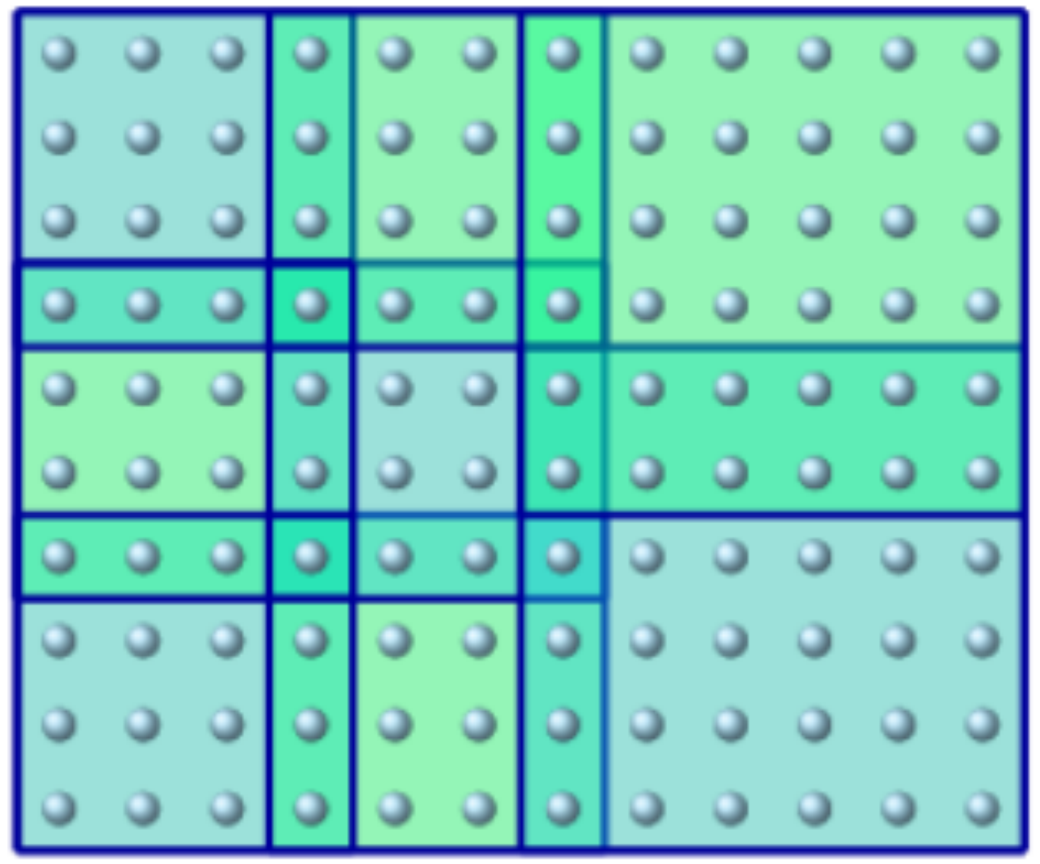
- Hilbert space associated with Λ : $\mathscr{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{H}_x \equiv \bigotimes_{x\in\Lambda} \mathbb{C}^d$

- Density matrices: $\mathscr{S}_\Lambda := \mathscr{S}(\mathscr{H}_\Lambda) = \{\rho \in \mathscr{B}(\mathscr{H}_\Lambda) \ : \ \rho \geq 0, \ \mathrm{tr}[\rho] = 1\}$



- Hamiltonian: $H_\Lambda = \sum_{X\subset\Lambda} H_X$

- Finite-range (k-local interactions): $\begin{cases} H_X = 0 \text{ for } \mathrm{diam}(X) > k \\ \|H_X\| < J \ \ \forall X \subset \Lambda \end{cases}$



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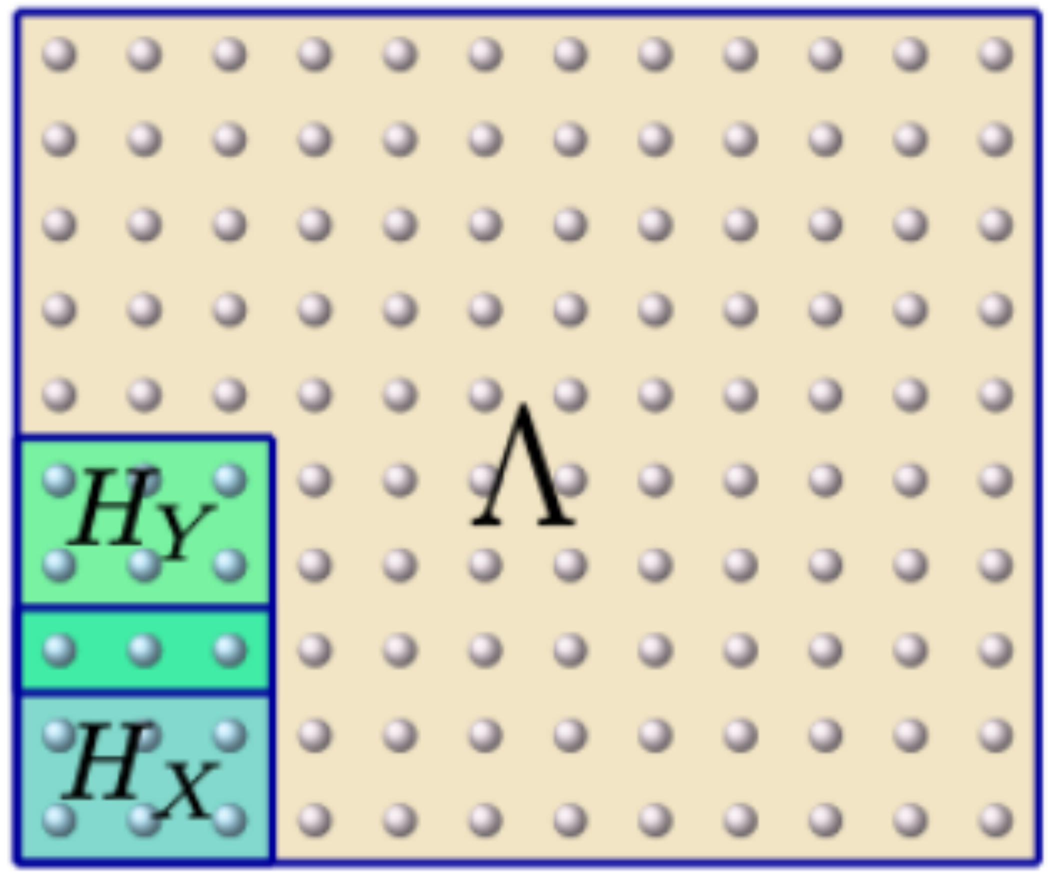

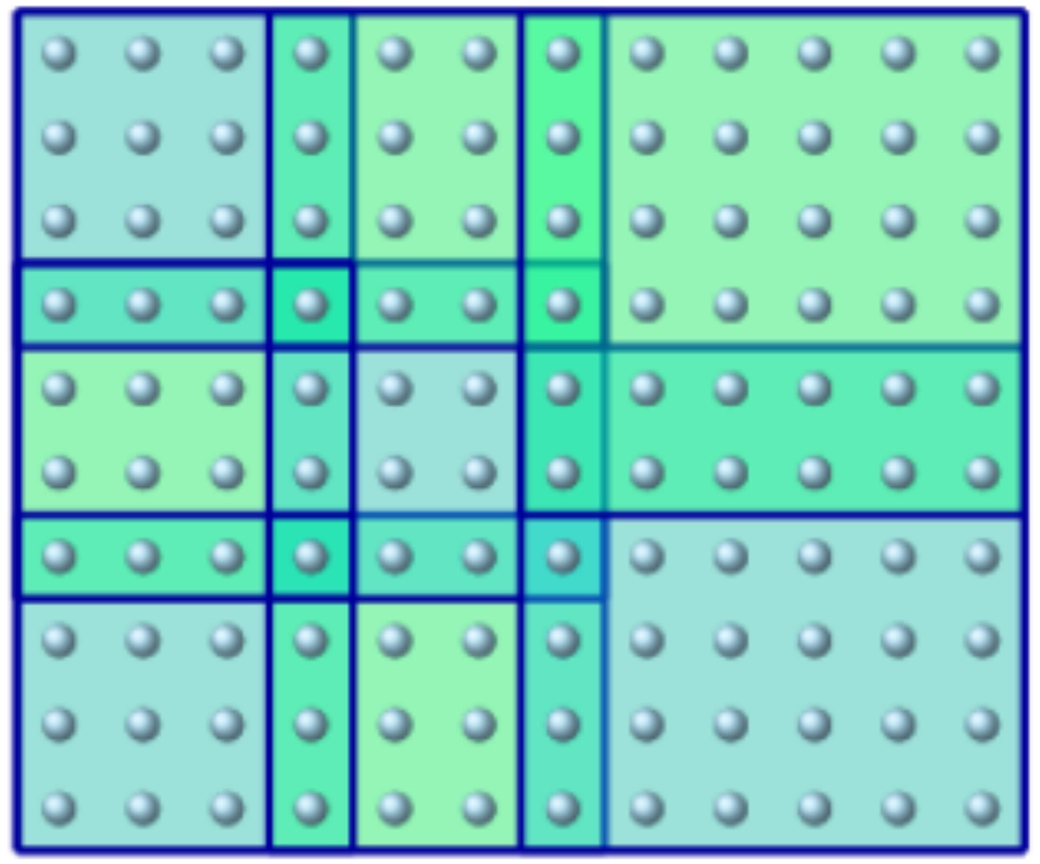

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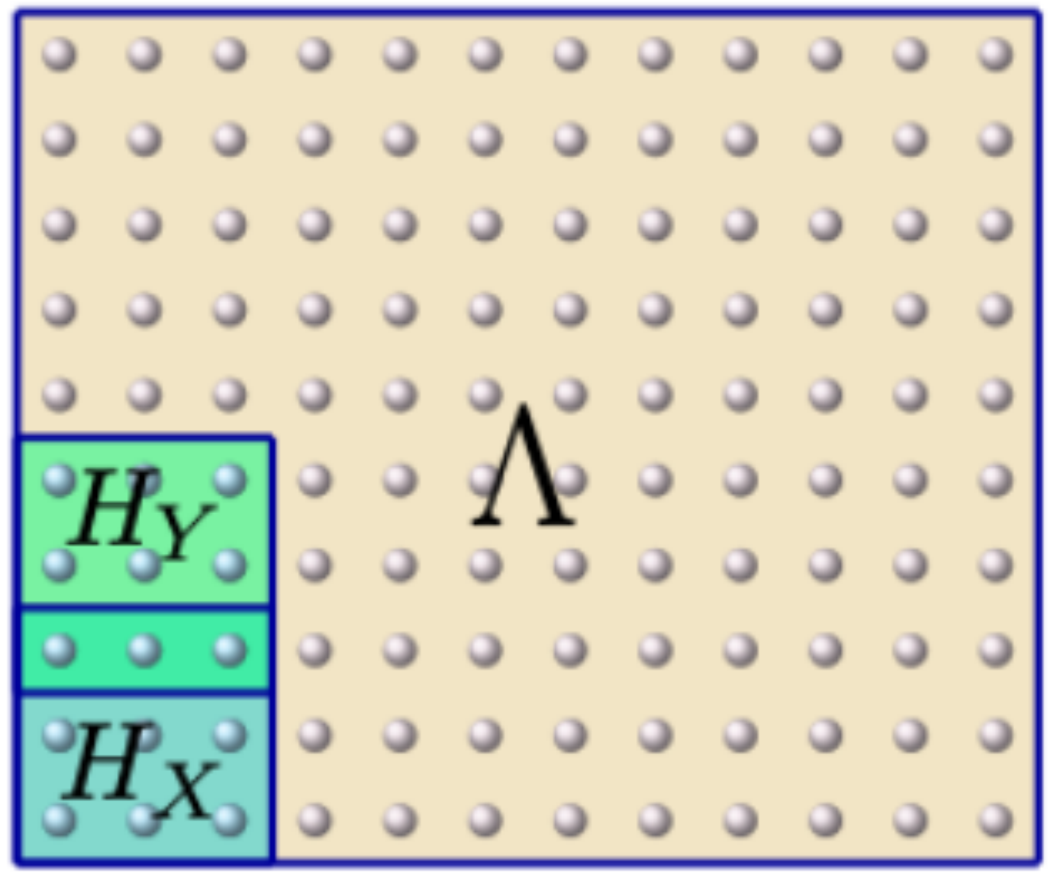
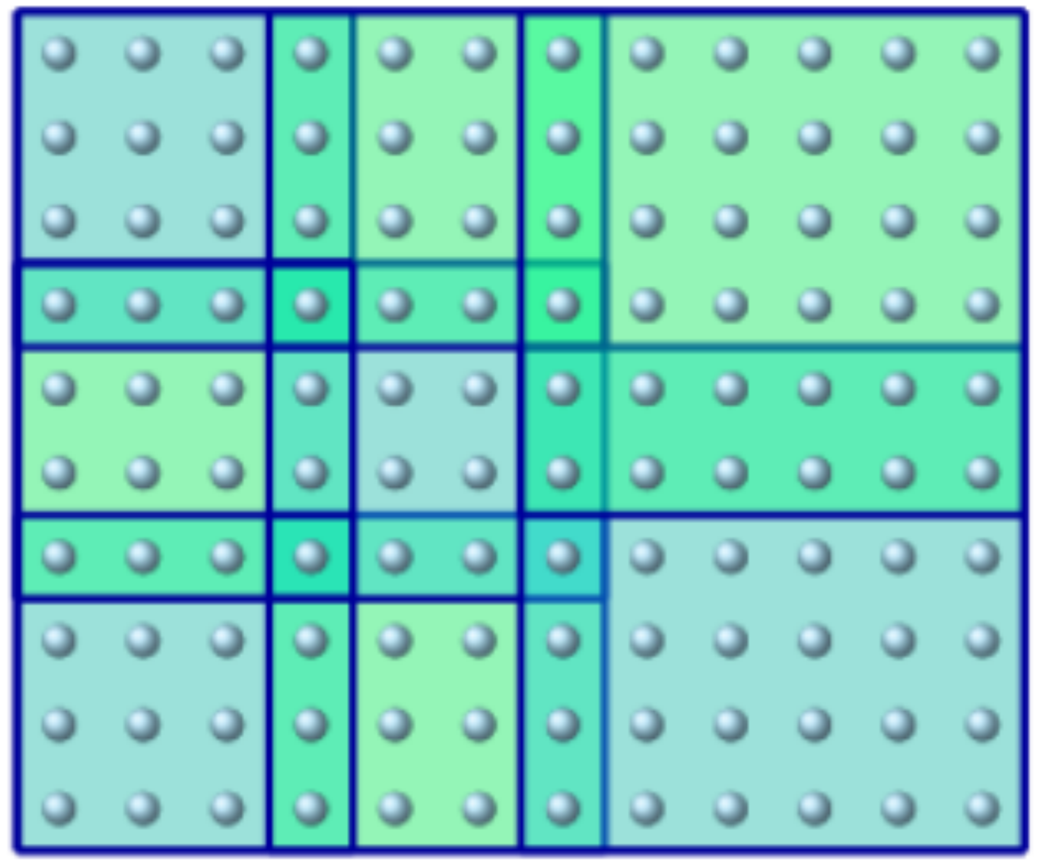
- Commuting: $[H_X, H_Y] = 0 \ \ \forall X, Y \subset \Lambda$
- Gibbs state (at inverse temperature $\beta > 0$): $\rho^\Lambda := \dfrac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}$





GIBBS SAMPLING / PREPARATION OF GIBBS STATES

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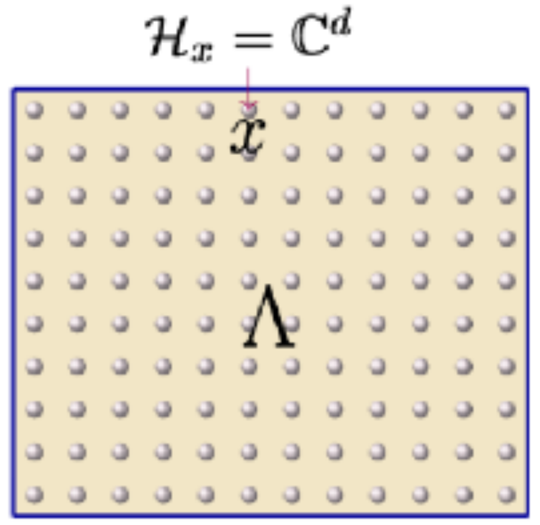

$$\sigma \rightarrow \boxed{\text{Gibbs Sampler}} \rightarrow \widetilde{\rho}$$

$$(\,\|\widetilde{\rho} - \rho\| \leq \varepsilon\,)$$

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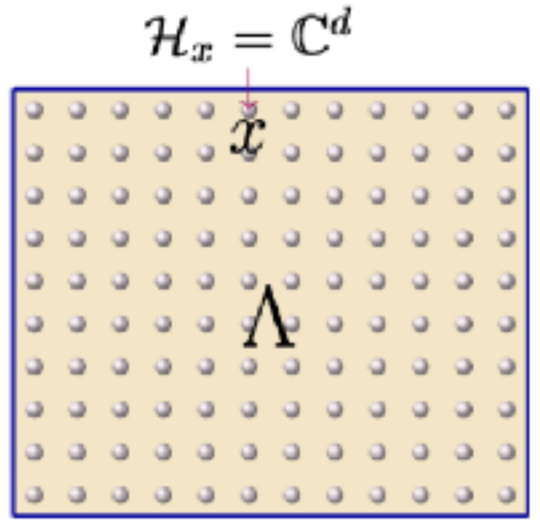

σ → **Gibbs Sampler** → $\tilde{\rho}$ $\quad (\|\tilde{\rho} - \rho\| \leq \varepsilon)$

How do we do **Gibbs sampling?**

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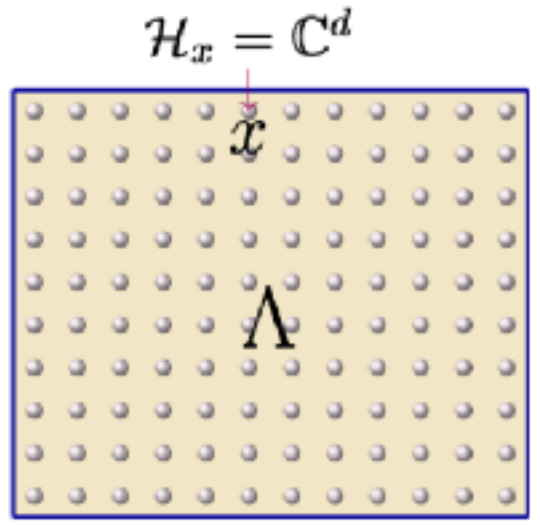

How do we do **Gibbs sampling?**

- A typical way is via **dissipation.**

Efficient Preparation of the Gibbs State of the 2D Toric Code

Via Dissipation

Modified logarithmic Sobolev inequalities for CSS codes

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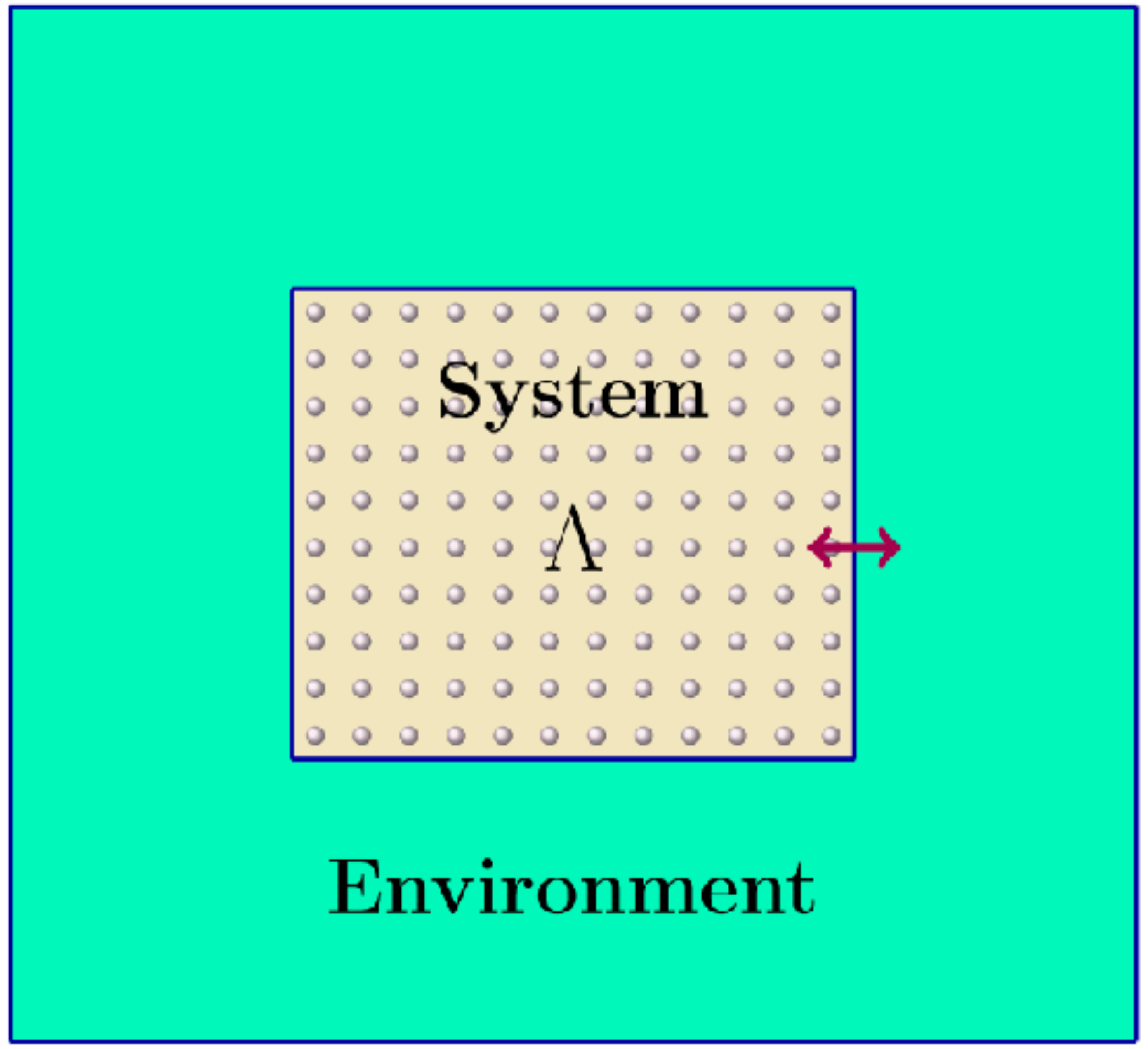

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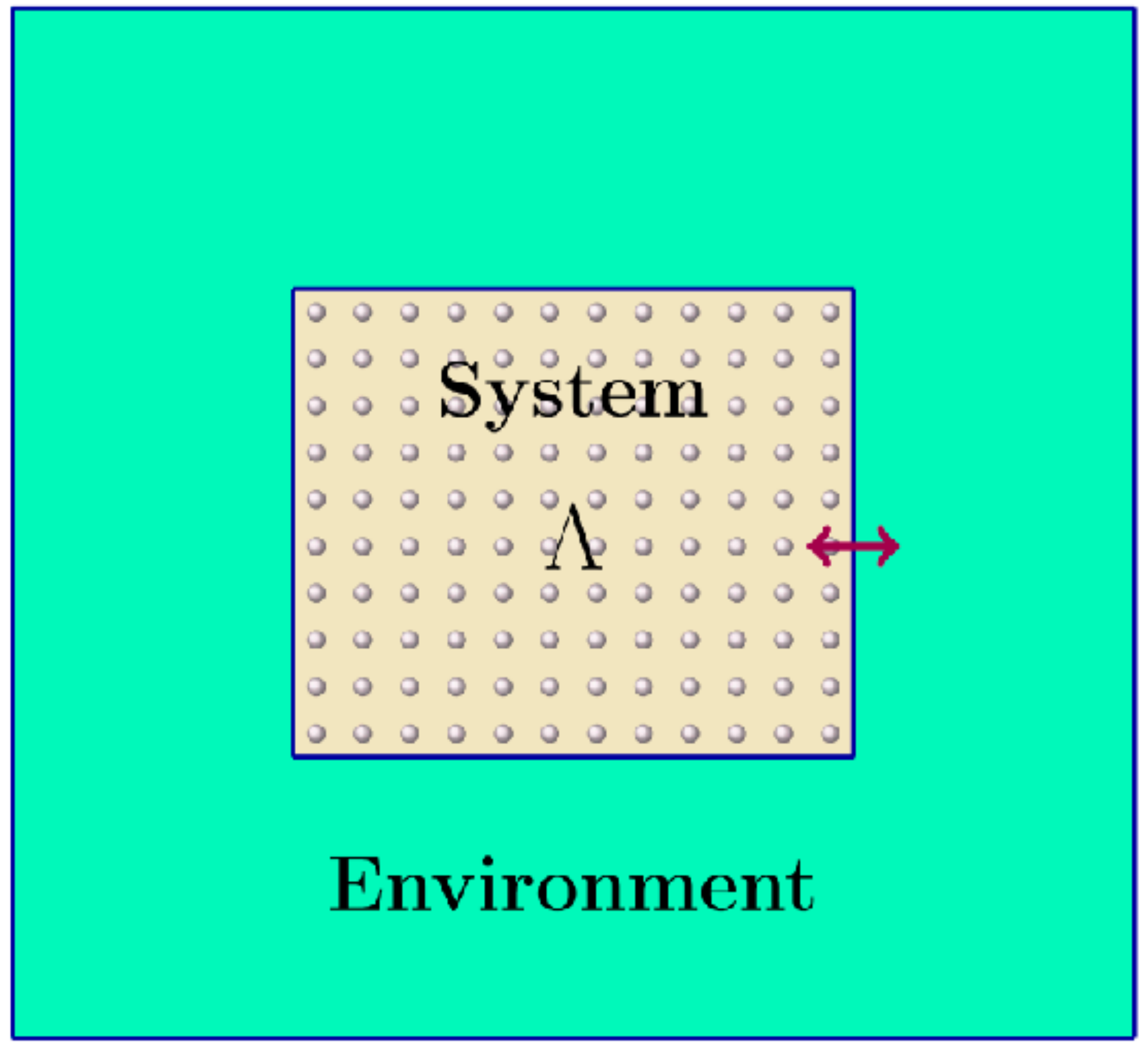

- The dynamics of the system is **dissipative**!
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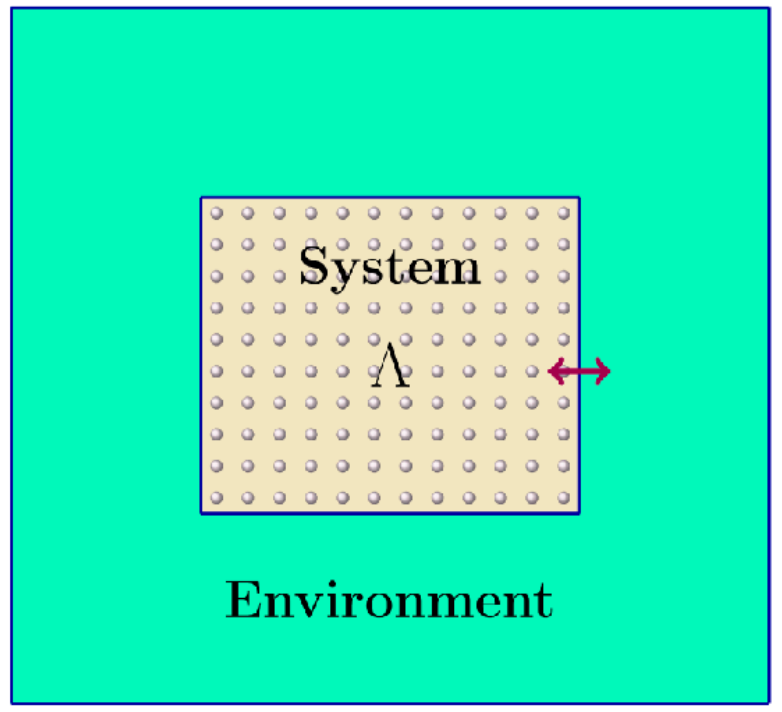

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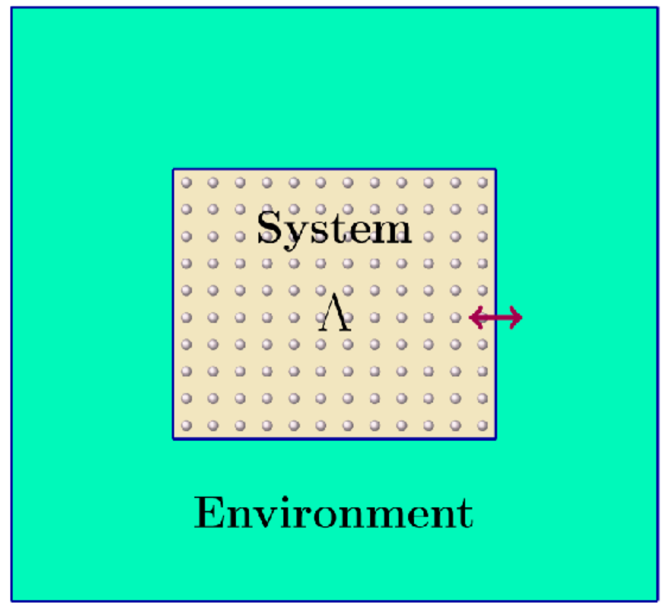

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- **Dissipative quantum state engineering:** Robust way of engineering relevant quantum states and algorithms

[Verstraete, Wolf, Cirac, NatPhys '09] [Kraus, Büchler, Diehl, Kantian, Micheli, Zoller, PRA '08]

EFFICIENT GIBBS SAMPLING WITH DISSIPATION

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Efficient preparation of Gibbs states

When do we have $\| \mathrm{e}^{t\mathcal{L}}(\sigma) - \rho \|_1 \leq \varepsilon$?

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EFFICIENT IMPLEMENTATION OF THE LINDBLADIAN

$$\boxed{e^{t\mathcal{L}}(\sigma) \xrightarrow{t\to\infty} \rho}$$

1. **Commuting case:** Efficient implementation of Davies generator

   [Rall, Wang, Wocjan, Quantum'23] [Li, Wang ICALP'23]

2. **Non-commuting case:** Efficient implementation of the CKG generator

   [Chen, Kastoryano, Gilyén, arXiv:2311.09207]

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Number of qubits: $|\Lambda|$

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RAPID/FAST MIXING OF THE EVOLUTION

$$\mathrm{e}^{t\mathcal{L}}(\sigma) \xrightarrow{t\to\infty} \rho$$

Modified logarithmic Sobolev inequality:

$$D(\mathrm{e}^{t\mathcal{L}}(\sigma)\|\rho) \leq D(\sigma\|\rho)\,\mathrm{e}^{-2\alpha(\mathcal{L})t}$$

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⬇

Rapid mixing:

$$\sup_{\sigma\in\mathcal{S}(\mathcal{H}_\Lambda)} \|e^{t\mathcal{L}}(\rho)-\sigma\|_1 \leq \mathrm{poly}(|\Lambda|)\,\mathrm{e}^{-\gamma t}$$

Mixing time: $\tau_{\mathrm{mix}}(\varepsilon) = \mathcal{O}(\mathrm{polylog}\,|\Lambda|)$

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Spectral gap

Fast mixing:

$$\sup_{\sigma\in\mathcal{S}(\mathcal{H}_\Lambda)} \|e^{t\mathcal{L}}(\rho)-\sigma\|_1 \leq \exp(|\Lambda|)\,\mathrm{e}^{-\gamma t}$$

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   - 1D, TI, any positive temperature, rapid mixing
     [Bardet, AC, Gao, Lucia, Pérez-García, Rouzé, CMP'23 and PRL'23]
   - High D, 2-local, under decay of correlations + gap, rapid mixing
     [Kochanowski, Alhambra, AC, Rouzé, CMP'25
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     [AC, Gondolf, Kochanowski, Rouzé, arXiv:2412.017322]
   - 2D, quantum double models, fast mixing
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2. Non-commuting case: Any dimension, high-enough temperature, rapid mixing
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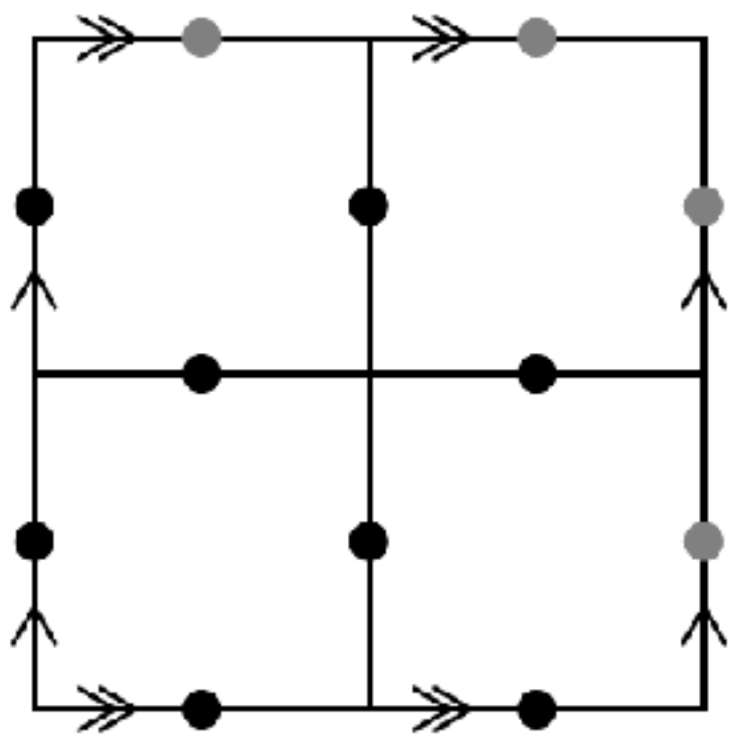
Can we prove rapid mixing for the

2D toric code and similar models?

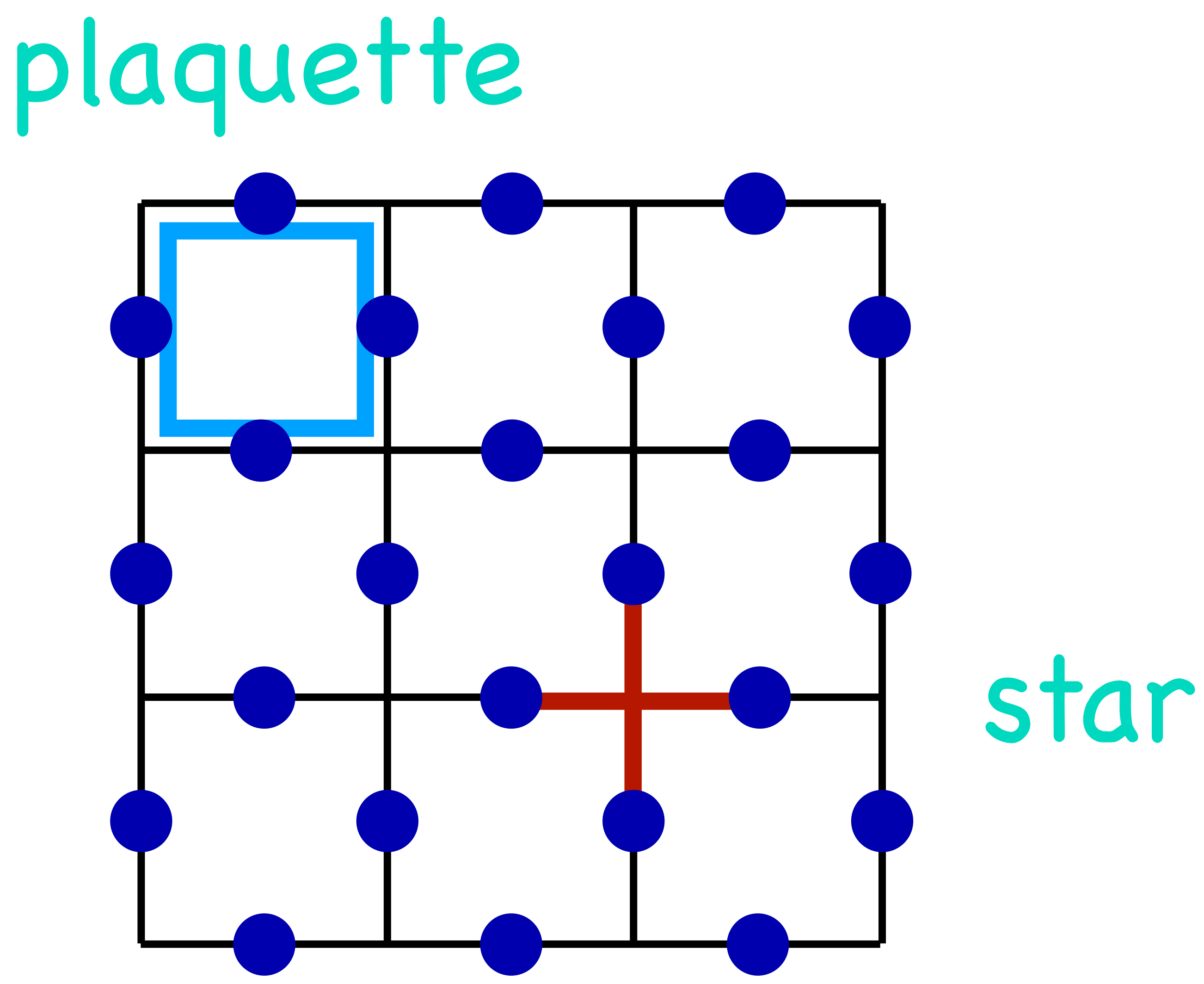
2D Toric code

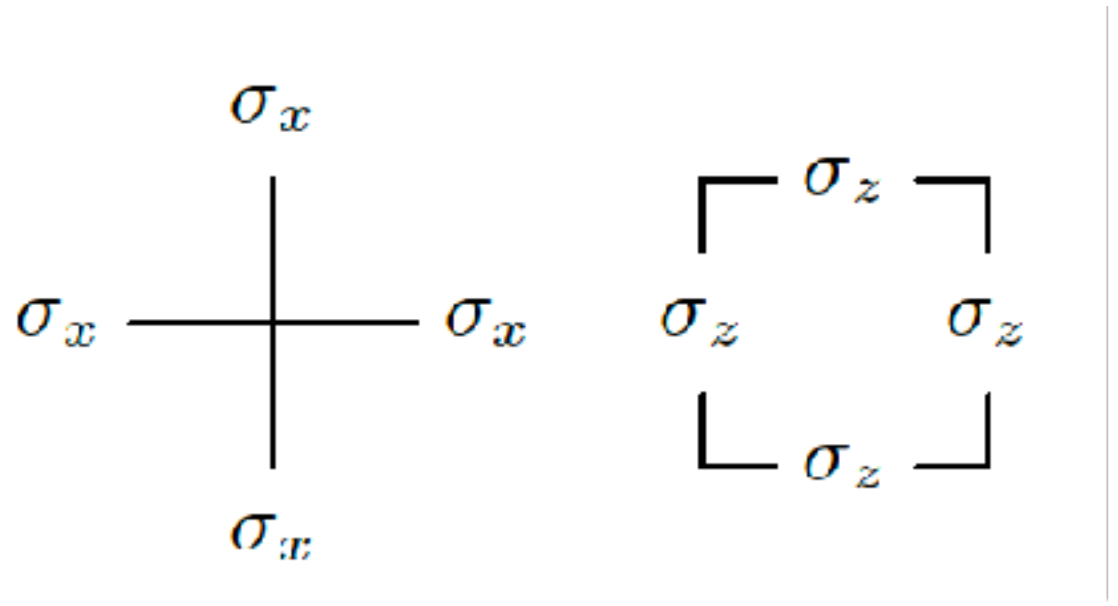
2D TORIC CODE

Geometry



Interactions





Hamiltonian

$$H_{TC} = -\sum_{s\in\mathbb{S}_\Lambda} J_v A_v - \sum_{p\in\mathbb{P}_\Lambda} J_p B_p \qquad A_v := \bigotimes_{i\in\partial v} \sigma_x^i, \quad B_p := \bigotimes_{i\in p} \sigma_z^i.$$

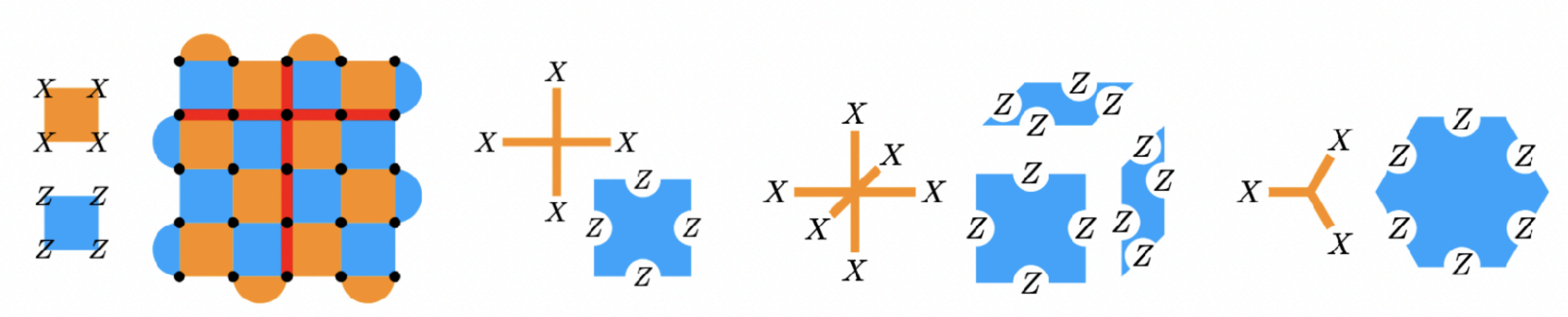
OTHER CSS CODES

ROTATED SURFACE CODE

2D TORIC CODE

3D TORIC CODE

TESSELLATION



Interactions

$$A_s := \bigotimes_{v\in ds} X_v \quad \text{and} \quad B_p := \bigotimes_{v\in\partial p} Z_v \qquad [A_s, B_p] = 0.$$

Hamiltonian

$$H_\Lambda^{\boxplus} := H_\Lambda^{\star} + H_\Lambda^{\square} \qquad H_\Lambda^{\star} := -\sum_{s\in\mathbb{S}_\Lambda} A_s, \qquad H_\Lambda^{\square} := -\sum_{p\in\mathbb{P}_\Lambda} B_p$$

RESULTS

2D Toric code The Davies Lindbladian associated to the 2D toric code has **rapid mixing** at every positive temperature

Loss of information in the 3D toric code

Since half of the Davies Lindbladian associated to the 3D toric code has rapid mixing at every positive temperature, **quantum information** in the 3D toric code is destroyed **exponentially fast**, and only classical information can survive long times

PREPARATION VIA DISSIPATION: LIMITATIONS OF THE APPROACH

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to prepare the Gibbs state

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Caveat: The mixing time depends exponentially on β!

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Next, we explore another simpler approach for specific models

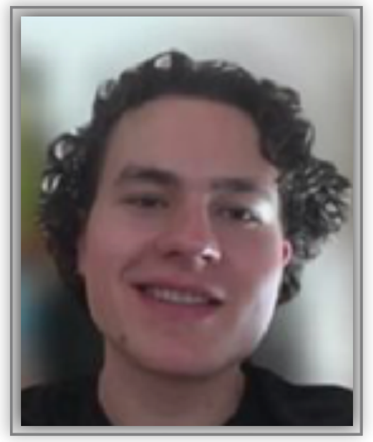
EFFICIENT PREPARATION OF THE GIBBS STATE OF THE 2D TORIC CODE

VIA DUALITY

Efficient and simple Gibbs state preparation of the 2D toric code via duality to classical Ising chains

arXiv:2508.00126

with

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DUALITY

Consider H_1 and H_2 two Hamiltonians.

We say that they are **poly-depth dual** if there exists a unitary U that can be implemented by a circuit (of 2-local gates) of polynomial depth such that

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Therefore, if ρ_1 can be **efficiently sampled**, ρ_2 as well.

DUALITY

Consider H_1 and H_2 two poly-depth dual Hamiltonians with

$$H_1 = UH_2U^\dagger \text{ and } \rho_1 = U\rho_2U^\dagger$$

Assume that ρ_1 can be efficiently sampled with $\mathscr{C}$.

Duality

Consider H_1 and H_2 two poly-depth dual Hamiltonians with

$$H_1 = UH_2U^\dagger \quad \text{and} \quad \rho_1 = U\rho_2U^\dagger$$

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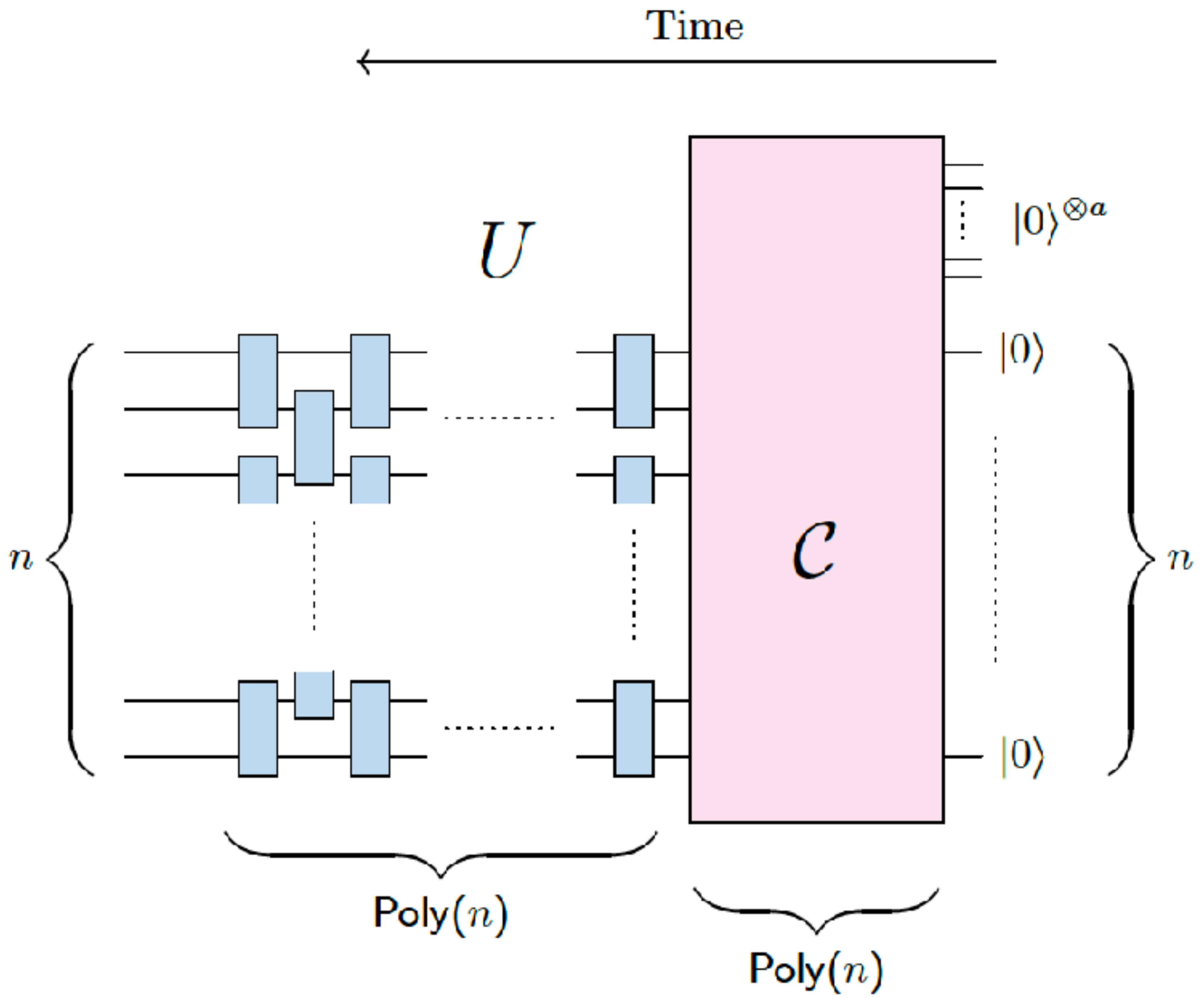
Then ρ_2 can be efficiently sampled with $U\mathscr{C}$.



QUANTUM GIBBS SAMPLING VIA DUALITY

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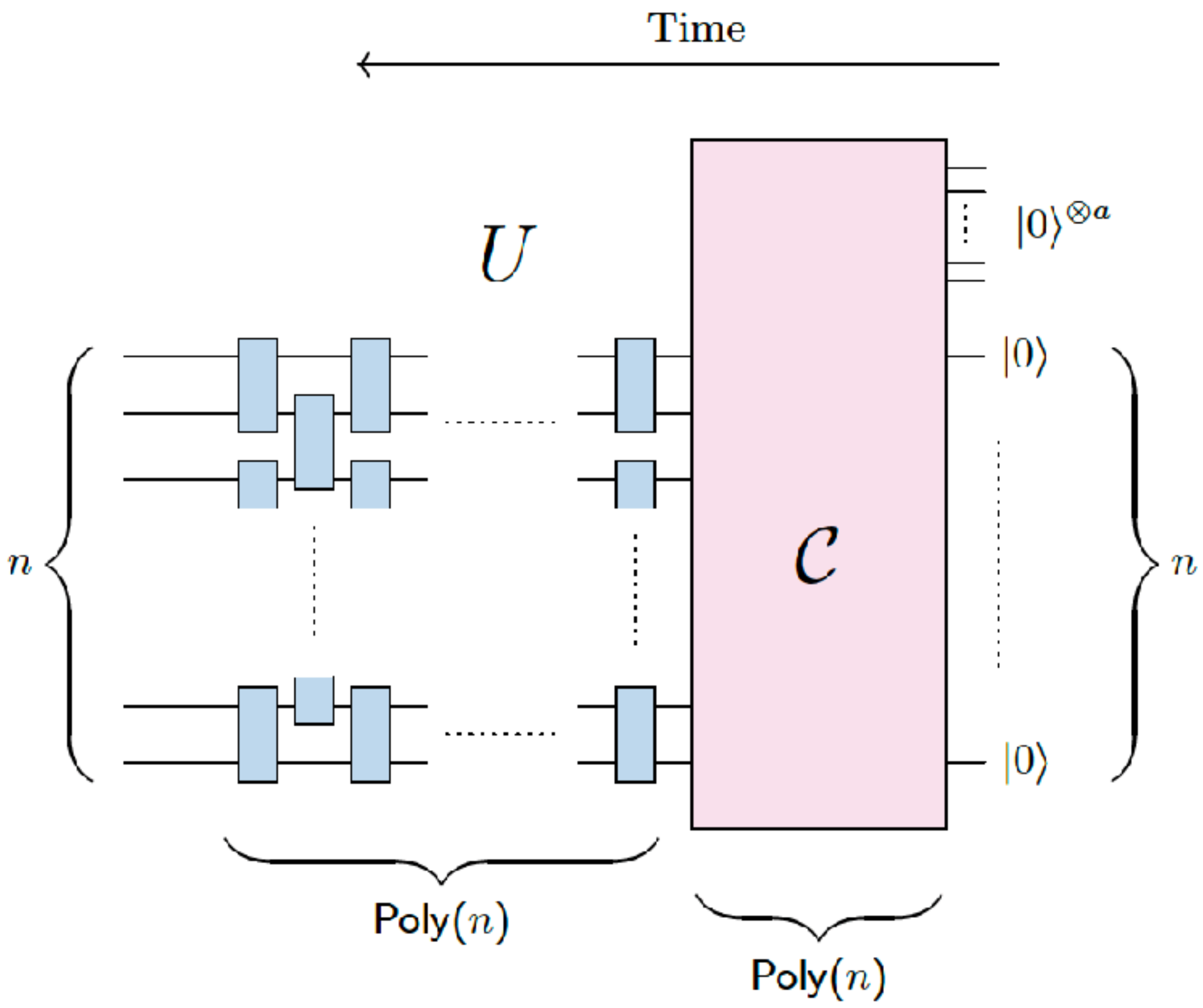


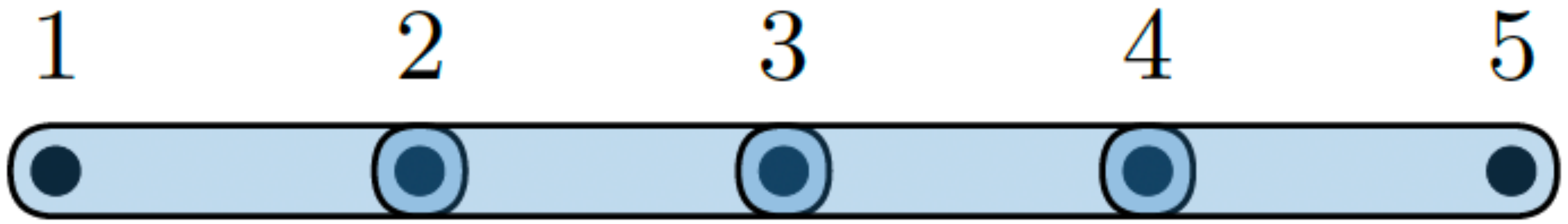
Ingredients. For a relevant Hamiltonian H_2:

1. Find a poly-depth circuit mapping it to H_1
2. Find an efficient sampler for ρ_1

EXAMPLE: 1D ISING CHAIN

CLASSICAL 1D ISING CHAIN (OF LENGTH L)

$$H = -\sum_{i=1}^{L-1} J_i \sigma_z^i \sigma_z^{i+1}$$



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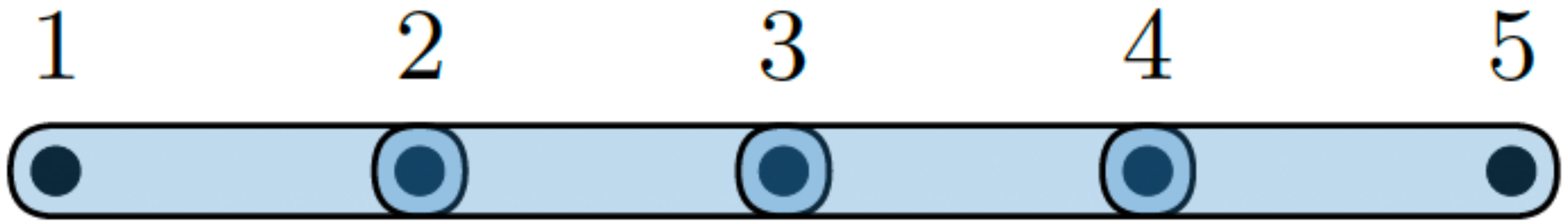

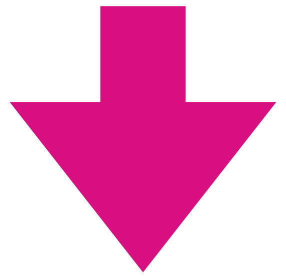

NON-INTERACTING HAMILTONIAN (OF LENGTH L)

$$UHU^\dagger = -\sum_{i=2}^{L} J_{i-1} \sigma_z^i$$

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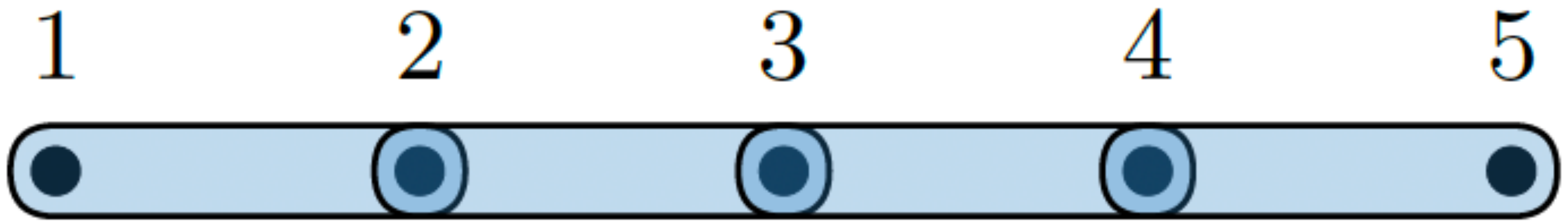

$$U := CX(1,2)\,CX(2,3)\cdots CX(L-1,L)$$

$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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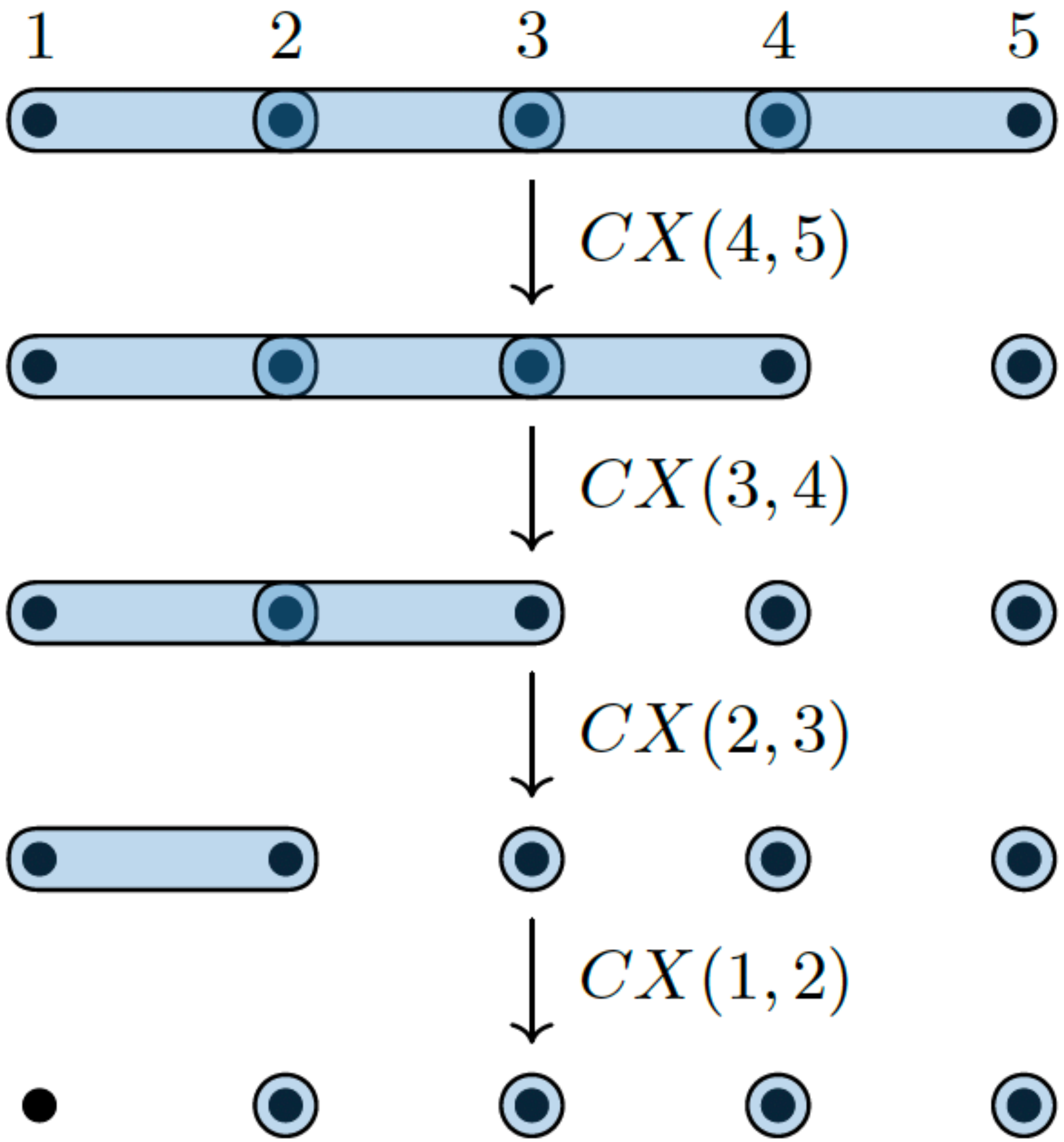

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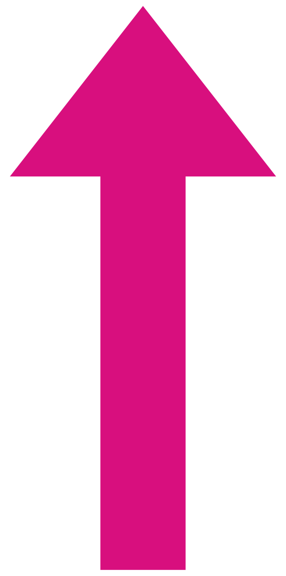

$\mathcal{O}(L)$ depth

NON-INTERACTING HAMILTONIAN

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$\dfrac{e^{-\beta UHU^\dagger}}{\mathrm{Tr}[e^{-\beta UHU^\dagger}]}$ can be sampled in $\mathcal{O}(1)$.

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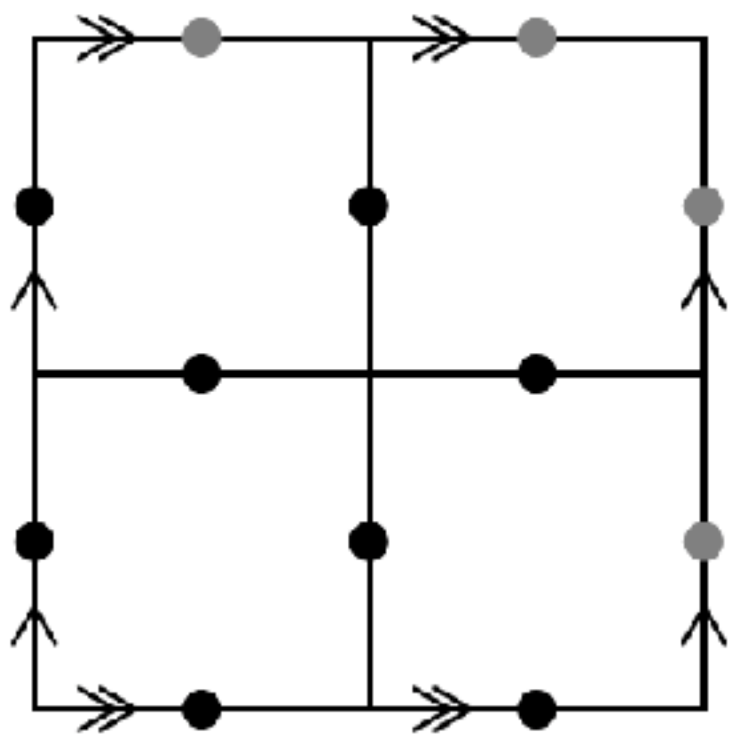
Duality between Toric code and Classical Ising chains

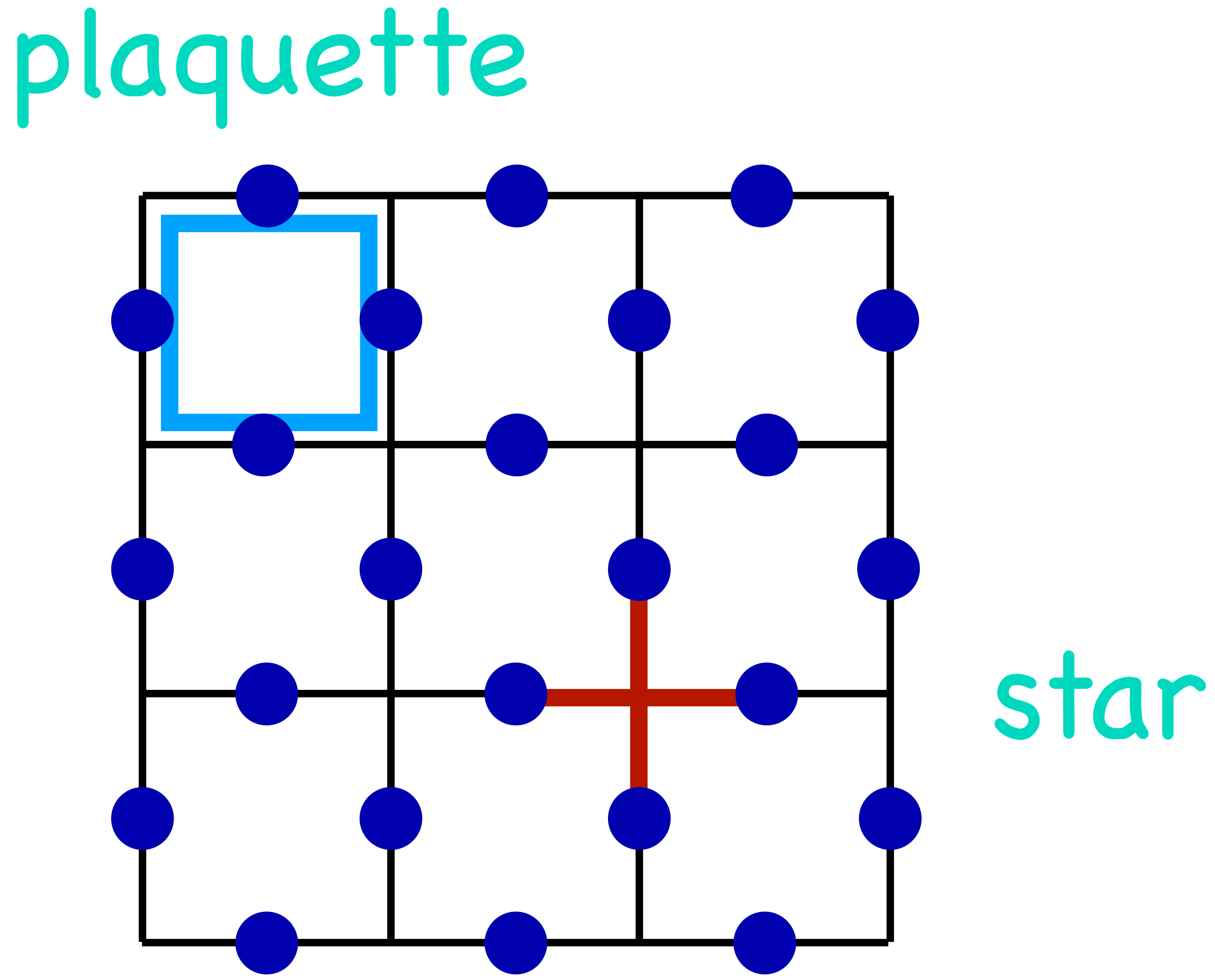
DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

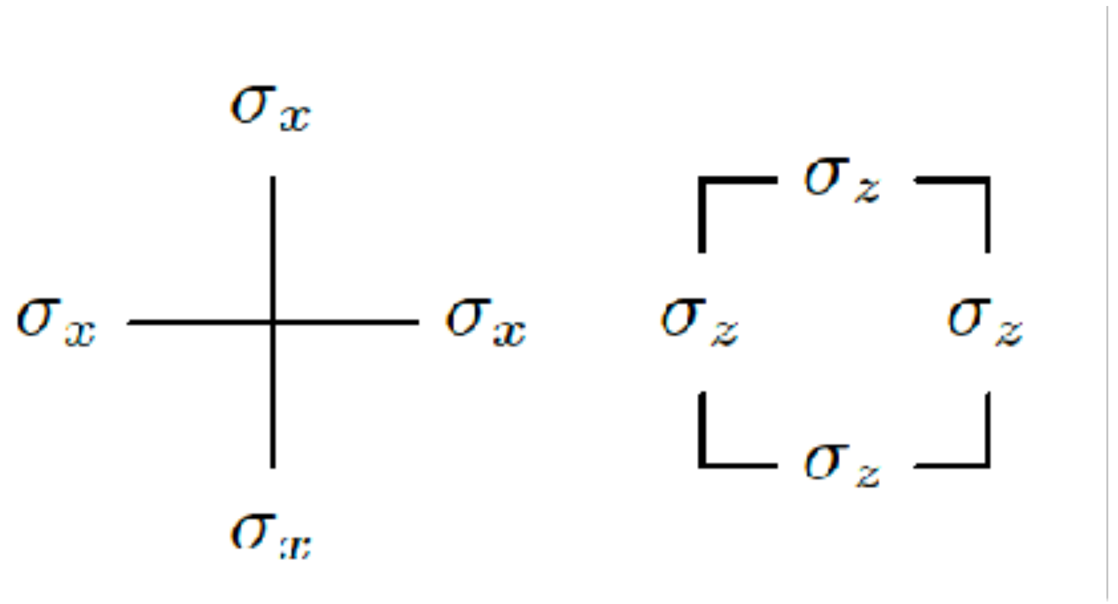
2D TORIC CODE

Geometry

Interactions







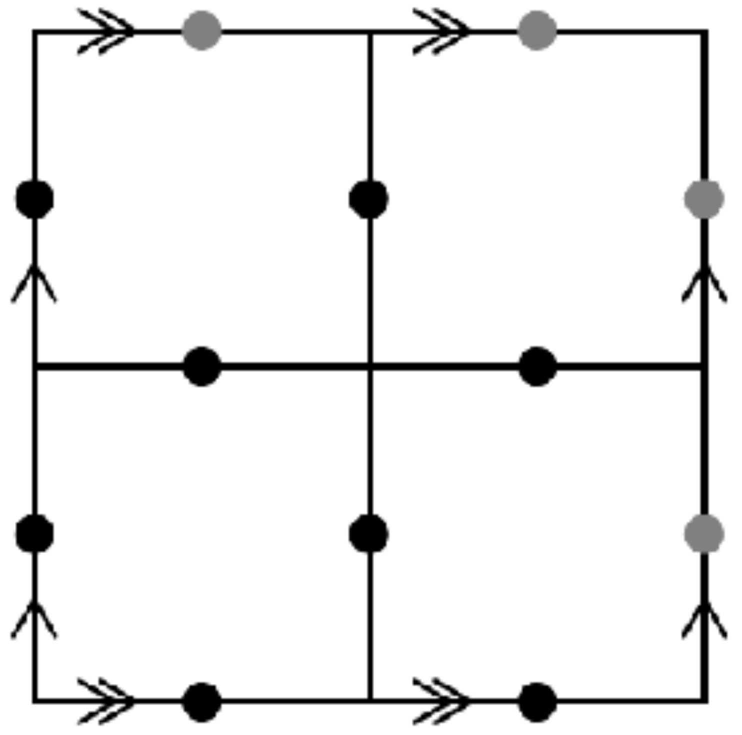
Hamiltonian

$$H_{TC} = -\sum_{v \in V_L} J_v A_v - \sum_{p \subset \mathcal{E}_L} J_p B_p \qquad A_v := \bigotimes_{i \in \partial v} \sigma_x^i, \quad B_p := \bigotimes_{i \in p} \sigma_z^i.$$

DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

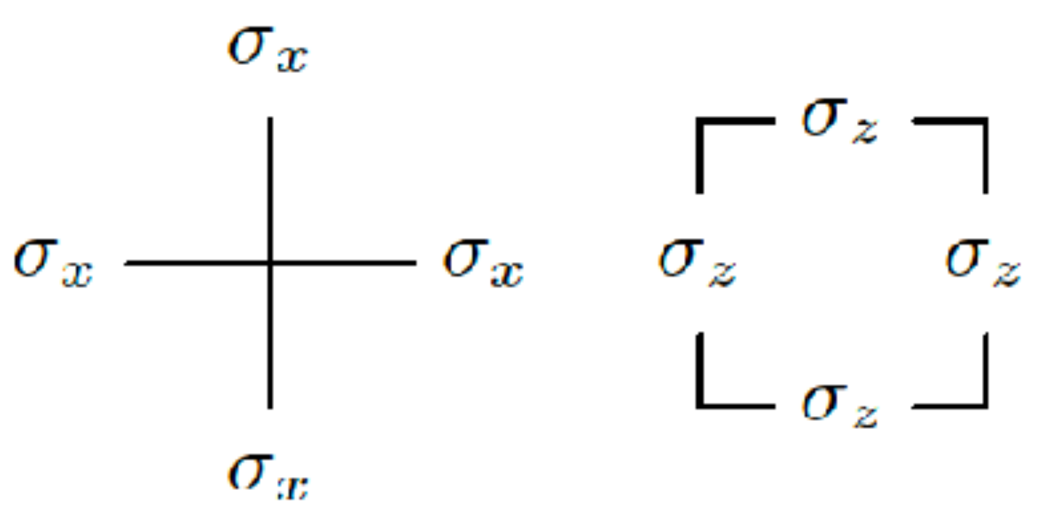
2D TORIC CODE

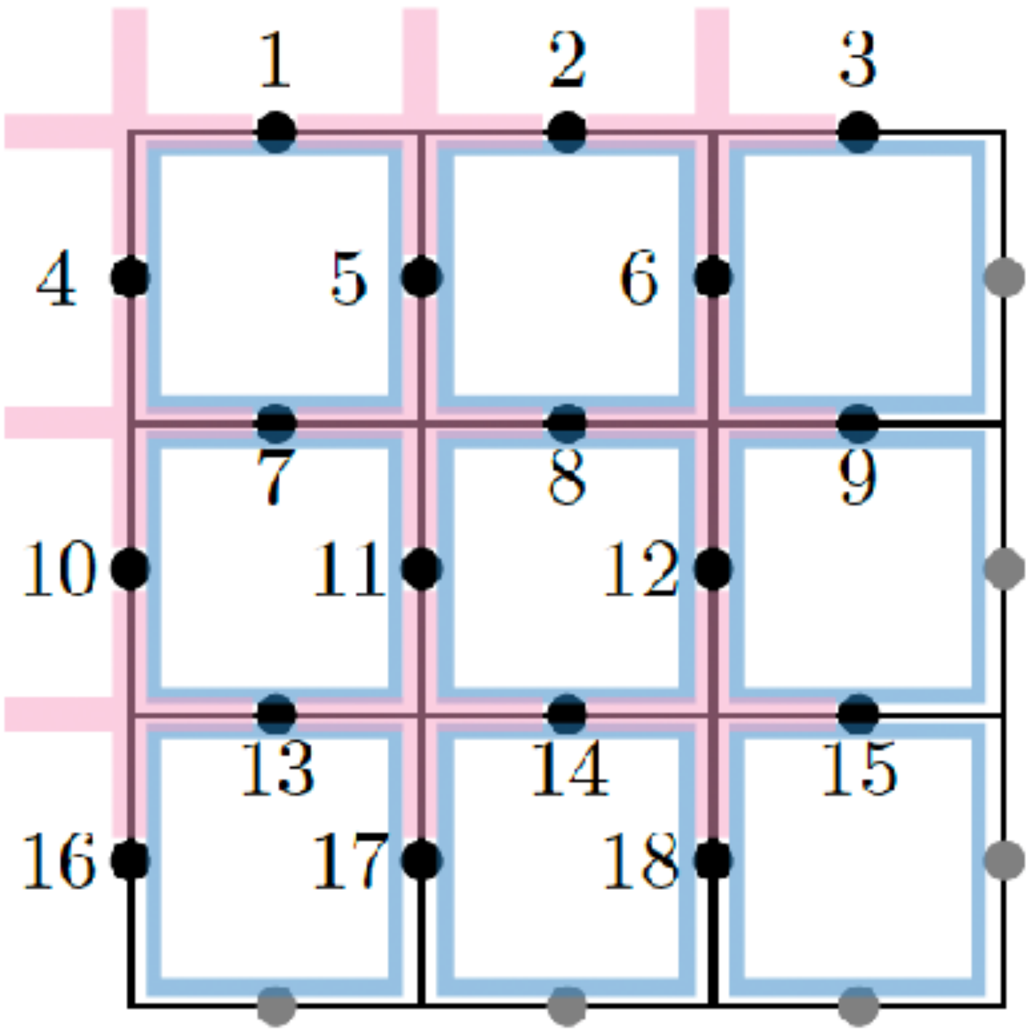
Geometry



Interactions

star plaquette





(for 3x3)

Hamiltonian

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DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

MAIN RESULT For the **2D Toric Code** in an $L \times L$ lattice, there exists a **quantum circuit** C composed of $\mathcal{O}(L^3)$ CX gates and $\mathcal{O}(L^2)$ Hadamard gates such that

$$C\Big(\sum_{v\in V_L} J_v A_v\Big)C^\dagger \text{ and } C\Big(\sum_{p\subset\mathcal{E}_L} J_p B_p\Big)C^\dagger$$

correspond to **2 disjoint 1D Ising chains.**

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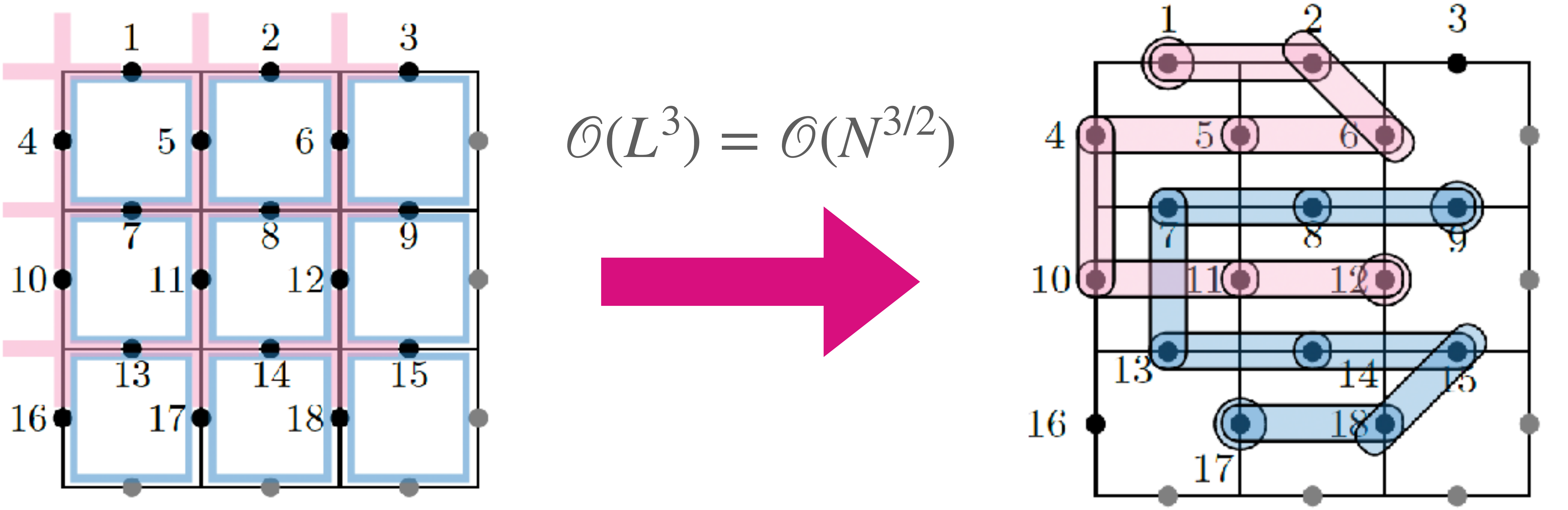

DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

MAIN RESULT

For the **2D Toric Code** in an $L \times L$ lattice,
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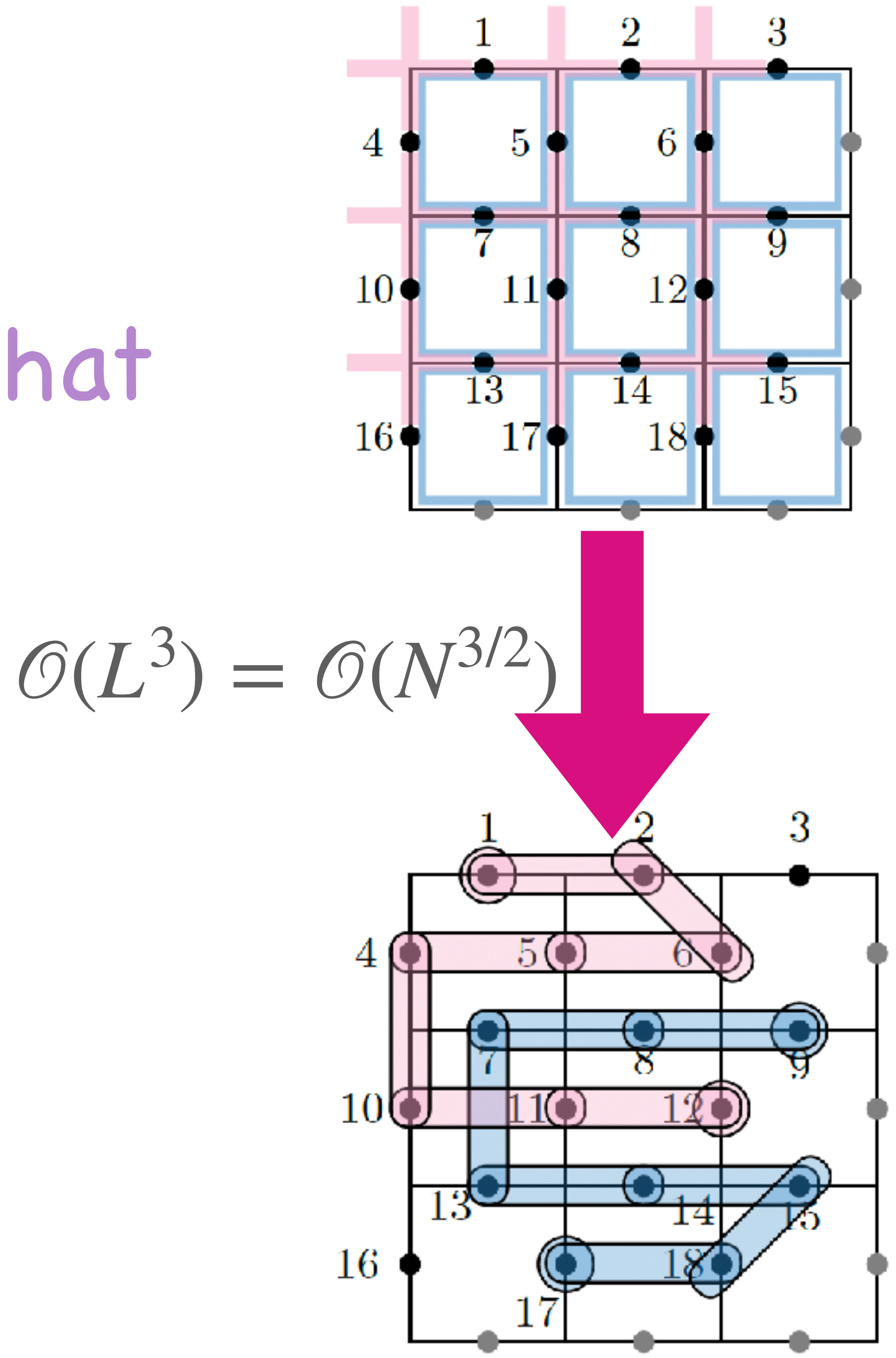

$$\mathcal{O}(L^3) = \mathcal{O}(N^{3/2})$$

CONSEQUENCE

The **ground** and **Gibbs state** of the 2D Toric Code can be prepared with a **gate complexity** of $\mathcal{O}(L^3)$ for any $0 \leq \beta \leq \infty$.

DUALITY OF OTHER CSS CODES

CSS CODE

Hamiltonian $$-\sum_{v\in V_L} J_v A_v - \sum_{p\subset \mathcal{E}_L} J_p B_p$$

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with more general geometries.

DUALITY OF OTHER CSS CODES

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with more general geometries.

Commuting Pauli operators

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

with $\{H_i\}$ a collection of mutually orthogonal Pauli strings.

DUALITY OF OTHER CSS CODES

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

Result

The $\{H_i\}$ can be simultaneously diagonalised with a quantum circuit of cuadratic depth.

[van den Berg, Temme, Quantum'20]

[Aaronson, Gottesman, PRA'04]

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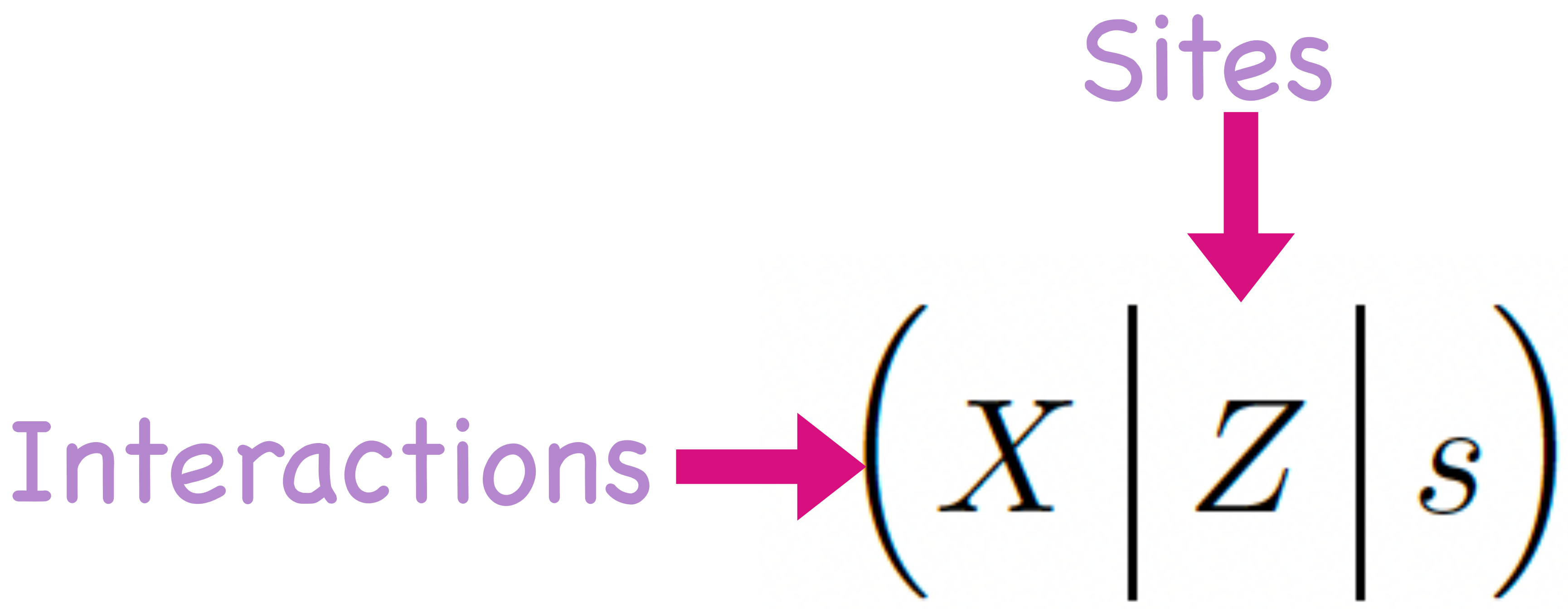
Idea of the proof

Write interactions of the Hamiltonian in a **tableau**:

Operator	x_{ij}	z_{ij}
σ_x	1	0
σ_z	0	1
σ_y	1	1
$\mathbb{1}$	0	0



$$\left(X \,\middle|\, Z \,\middle|\, s \right)$$

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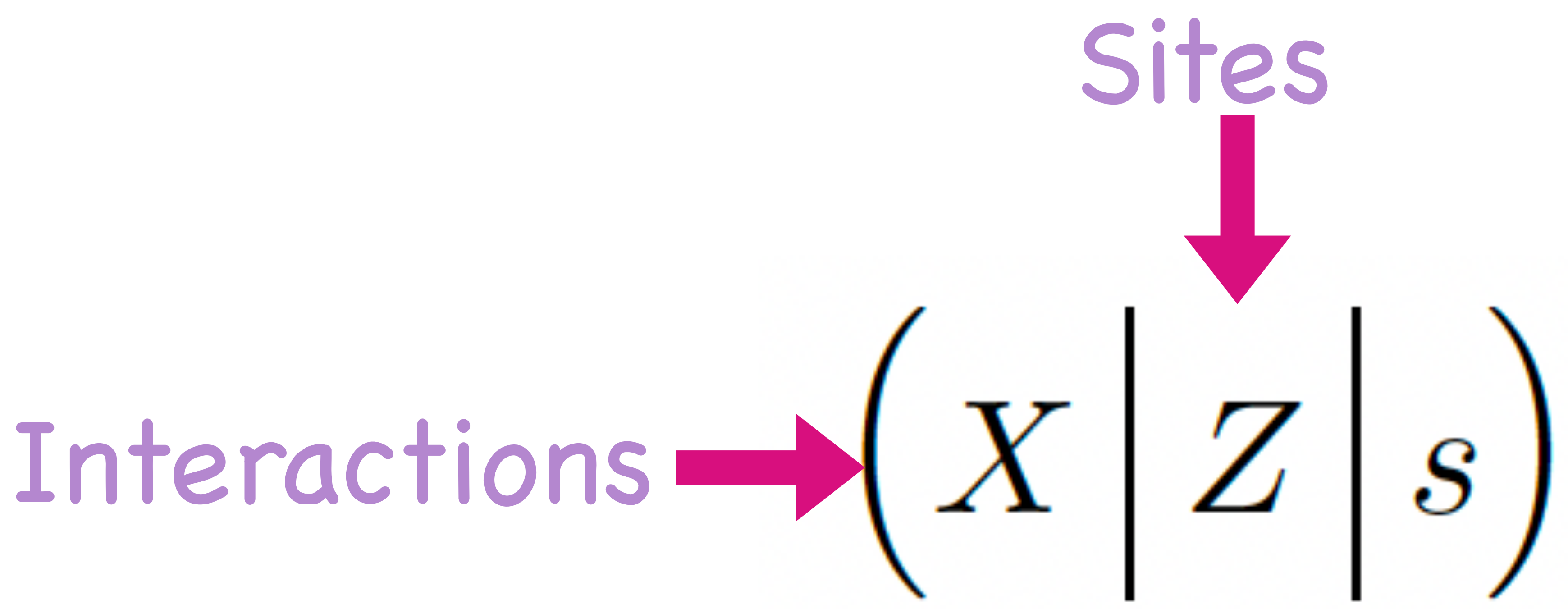

Sites

Interactions

$$\left(X \;\middle|\; Z \;\middle|\; s \right)$$

Example

$$\sigma_z \otimes \sigma_y \otimes \mathbb{1} - \sigma_x \otimes \mathbb{1} \otimes \sigma_y \rightarrow \left(\begin{array}{ccc|ccc|c} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

DUALITY OF OTHER CSS CODES

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Then, the aim is to reduce the X part of the matrix to all 0s and analyse the remaining Z part.

Duality of other CSS Codes

Result

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

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Idea of the proof

Write interactions of the Hamiltonian in a **tableau**:

Then, the aim is to reduce the X part of the matrix to all 0s and analyse the remaining Z part.

For these models, this is done with CX, Hadamard and Phase gates in $\mathcal{O}(n^2)$ depth.

DUALITY OF OTHER CSS CODES

Result

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

The $\{H_i\}$ can be simultaneously diagonalised with a quantum circuit of cuadratic depth.

These shows that all Hamiltonians composed of commuting Pauli operators are poly-depth dual to classical Hamiltonians.

Now the question is: To which classical Hamiltonians?

Duality of other CSS Codes

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

Example

If a tableau is achieved with Z part like

$$\left(\begin{array}{c|c|c}
\mathbf{I} & \mathbf{0} & 0\,0 \\
 & & \vdots \\
\hline
1\cdots1 & 0\cdots0 & \vdots \\
\hline
\mathbf{0} & \mathbf{I} & \vdots \\
\hline
0\cdots0 & 1\cdots1 & 0\,0
\end{array}\right)$$

these are two decoupled 1D Ising models and two spins without interactions.

DUALITY OF OTHER CSS CODES

Example

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$$\left(\begin{array}{c|c|c} \mathbf{I} & \mathbf{0} & 0\,0 \\ & & \vdots \\ \cline{1-2} 1\cdots1 & 0\cdots0 & \vdots \\ \cline{1-2} \mathbf{0} & \mathbf{I} & \vdots \\ \cline{1-2} 0\cdots0 & 1\cdots1 & 0\,0 \end{array}\right)$$

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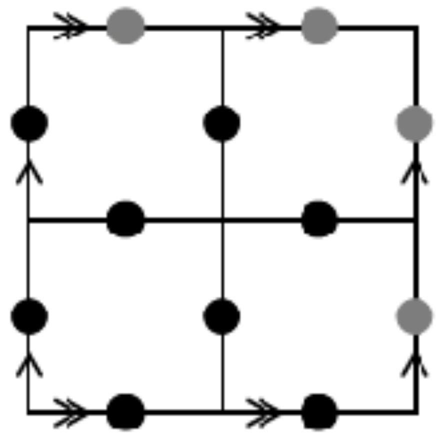
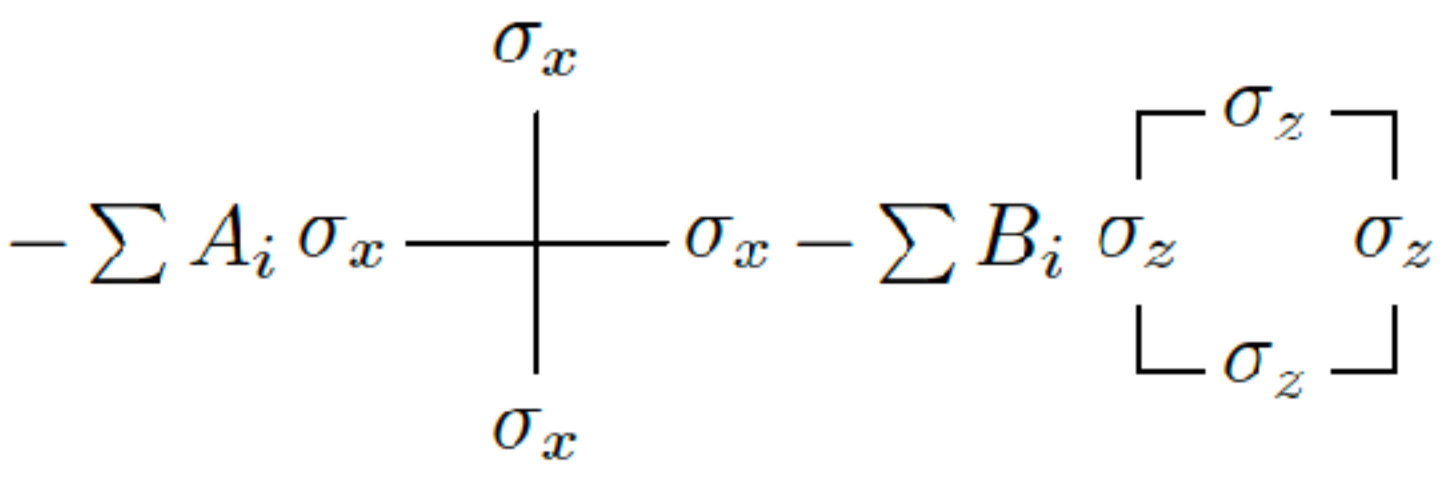
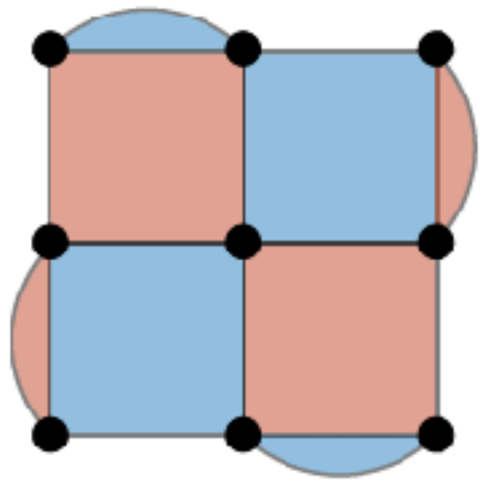
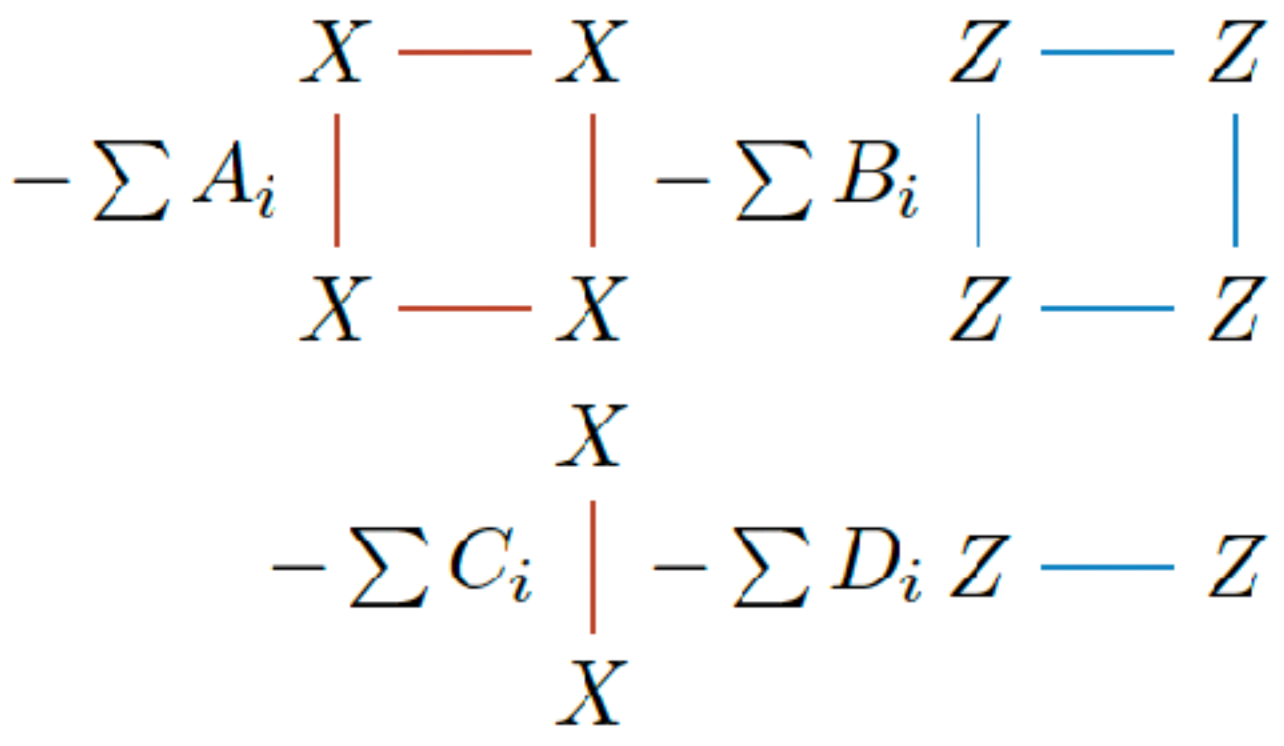
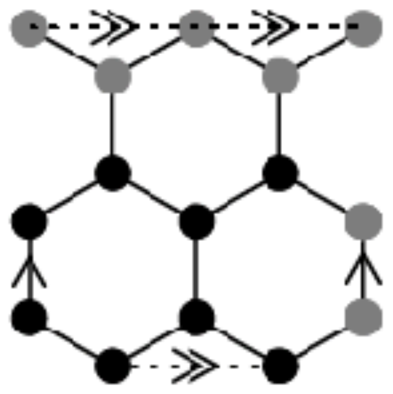
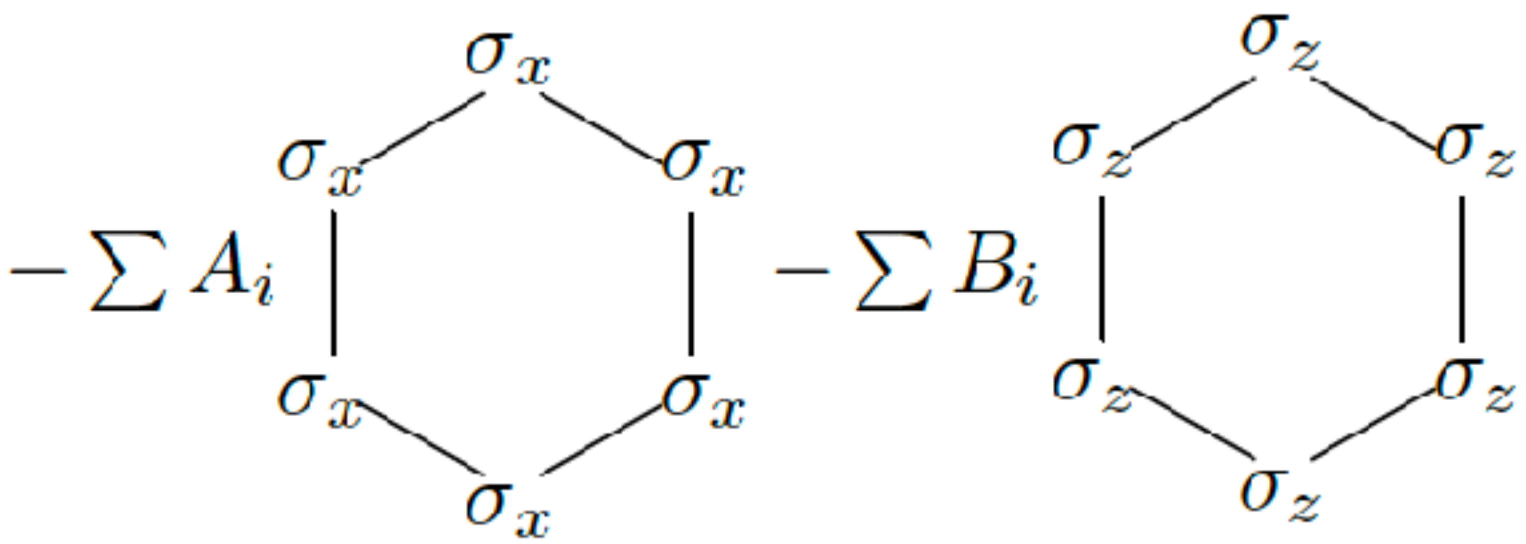
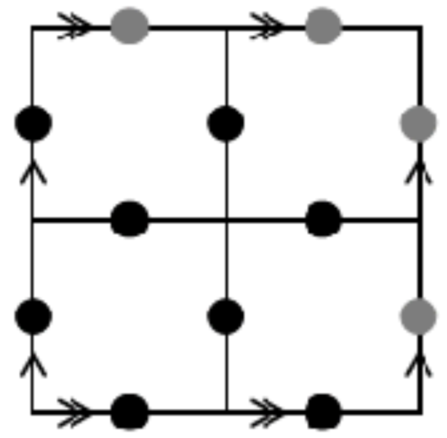
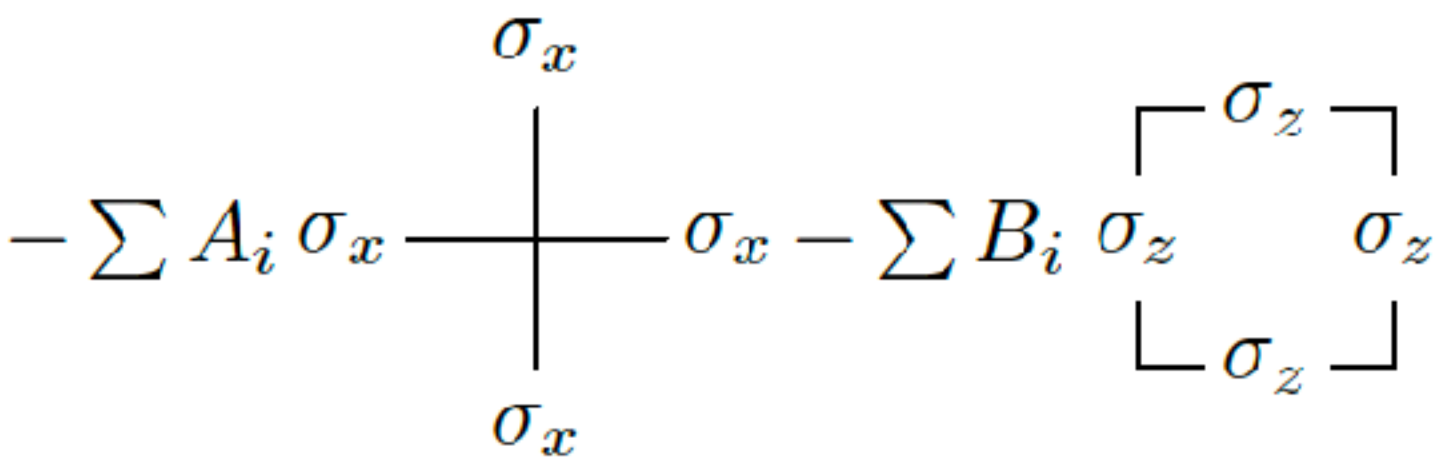
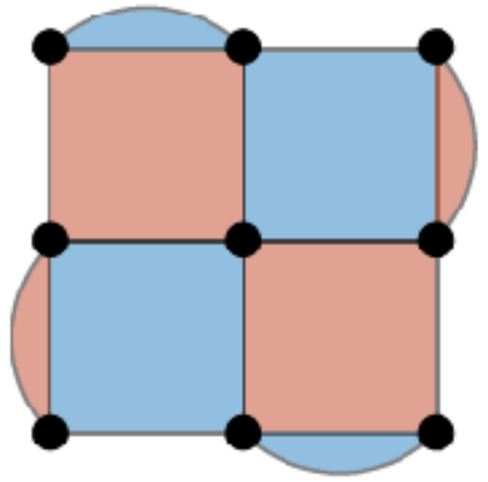
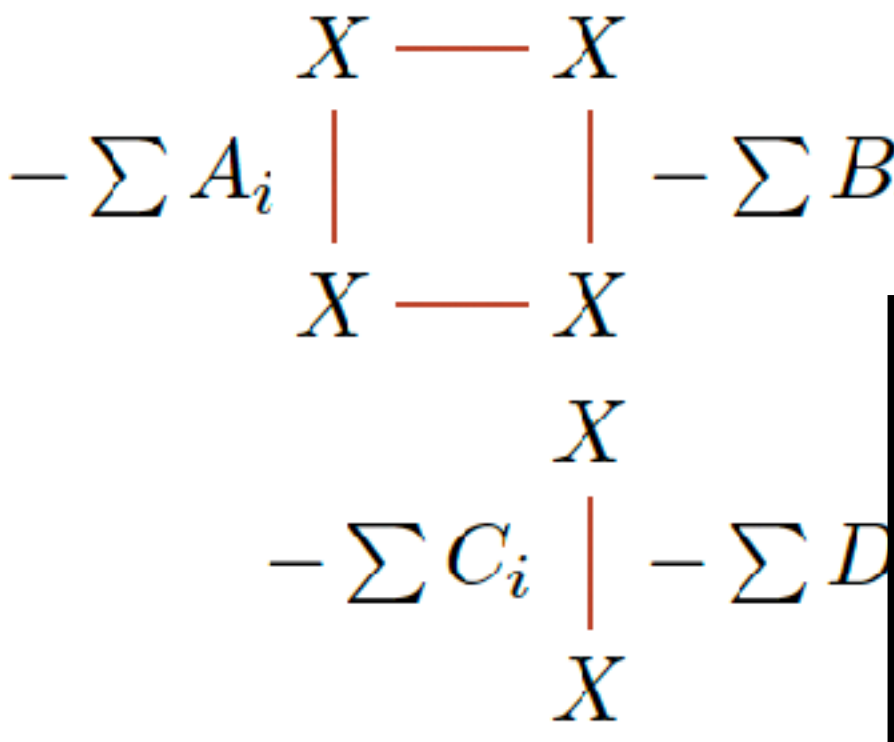
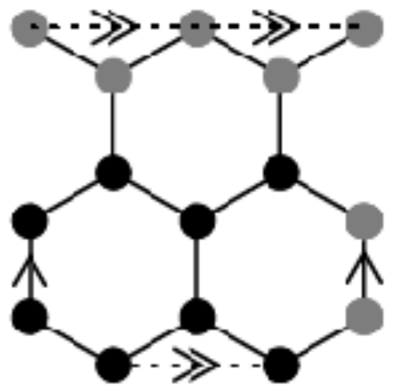
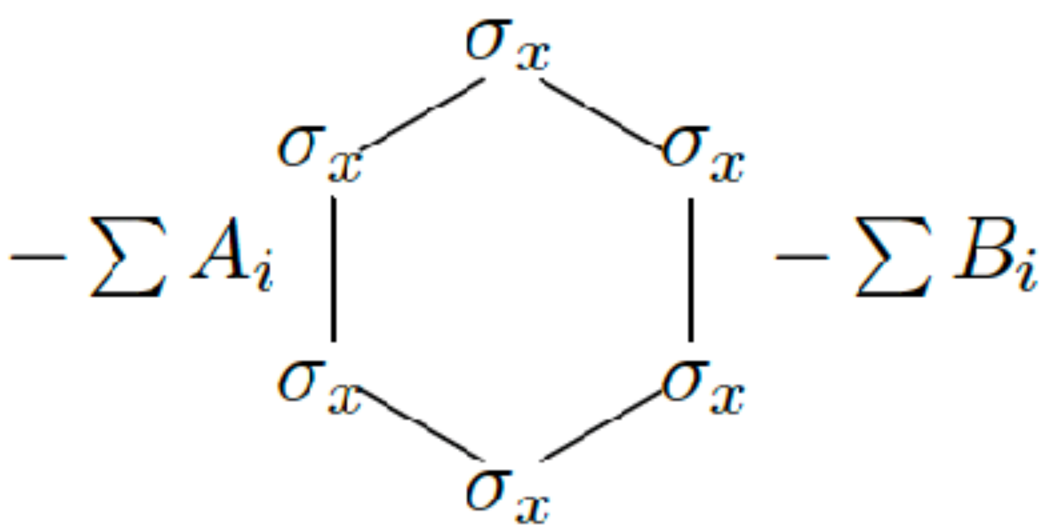
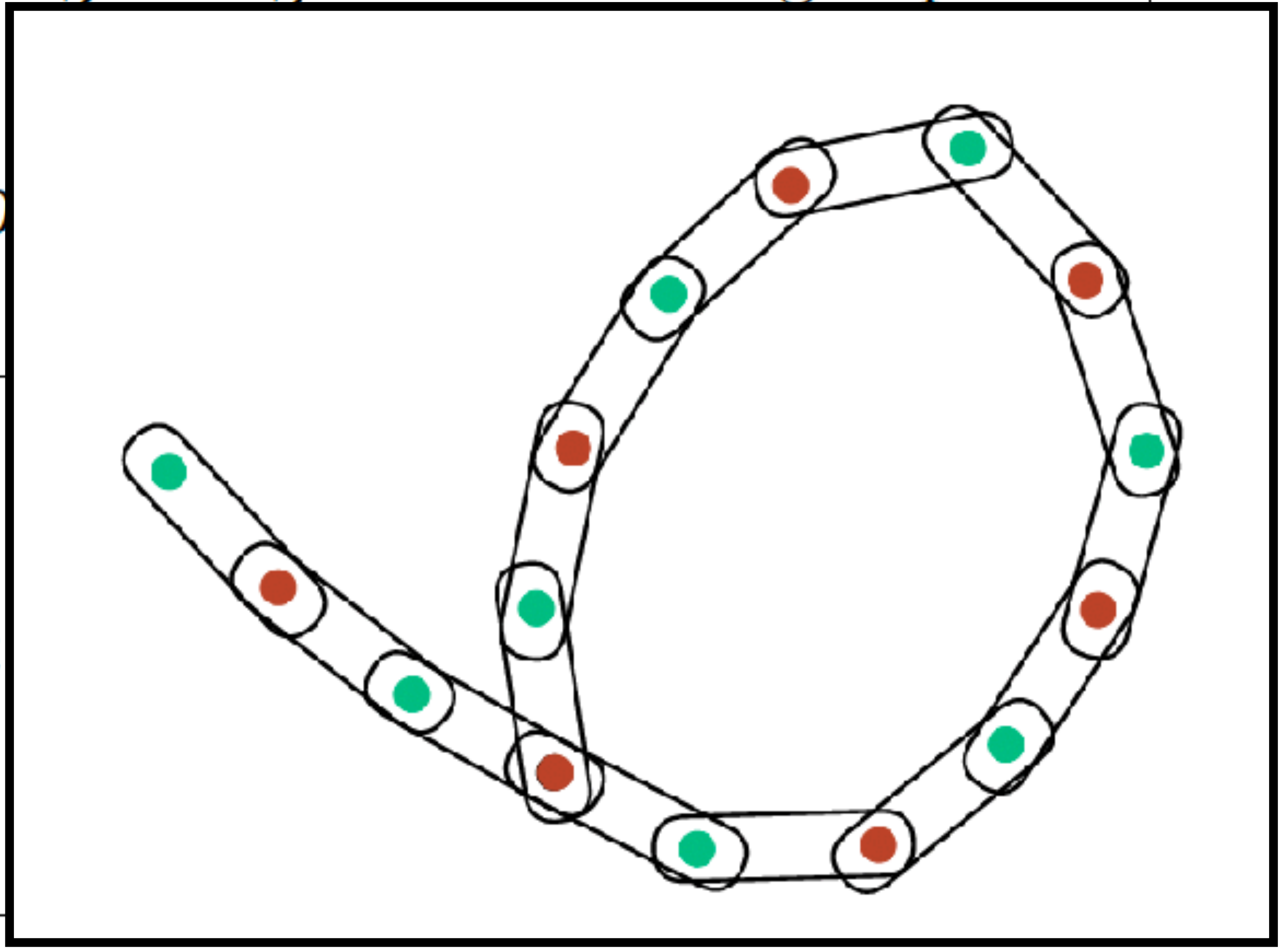
This is achieved from a 2D Toric Code.

DUALITY OF OTHER CSS CODES

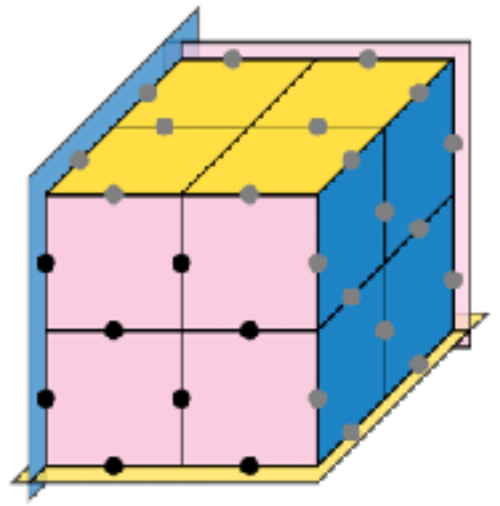
Original model	Lattice	Hamiltonian	Dual model	
2D toric code			Two decoupled Ising chains	Periodic boundary conditions
Rotated surface code			Non-interacting, single-spin Hamiltonian	Open boundary conditions
2D color code on a honeycomb lattice			Two decoupled lasso Ising chains if or non-interacting, single-spin Hamiltonian.	Periodic boundary conditions

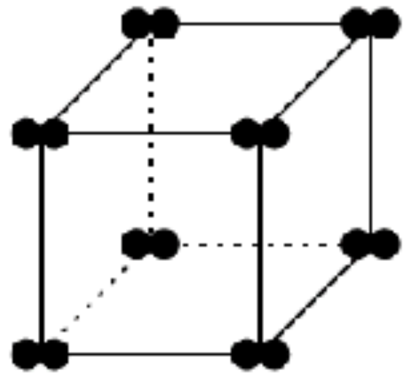
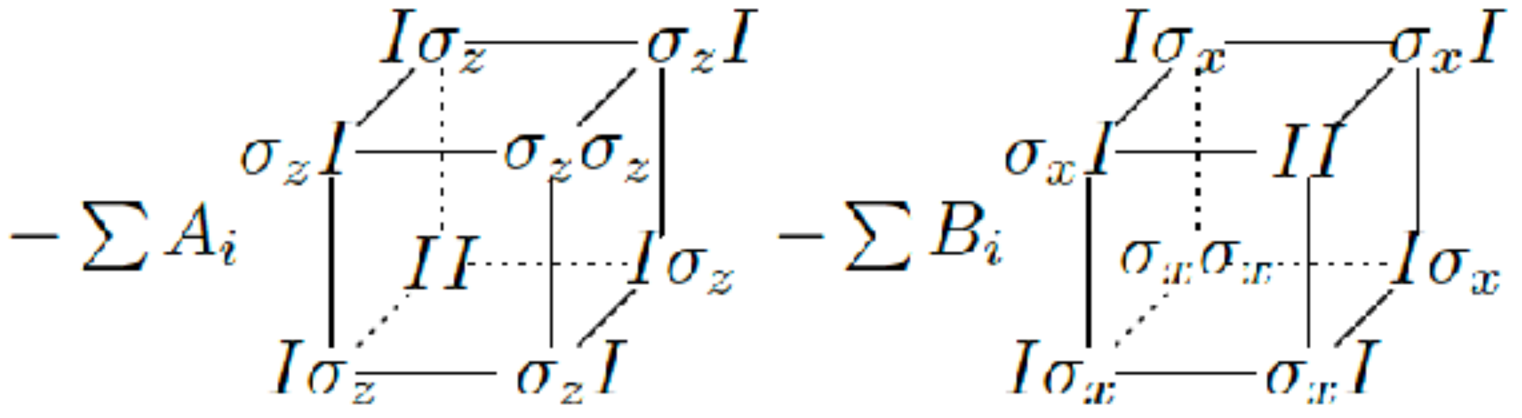
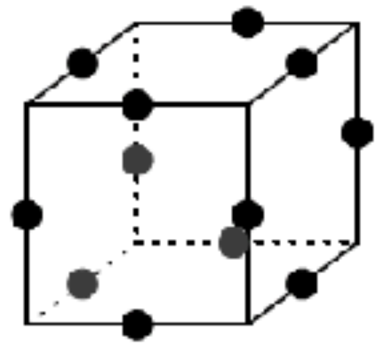
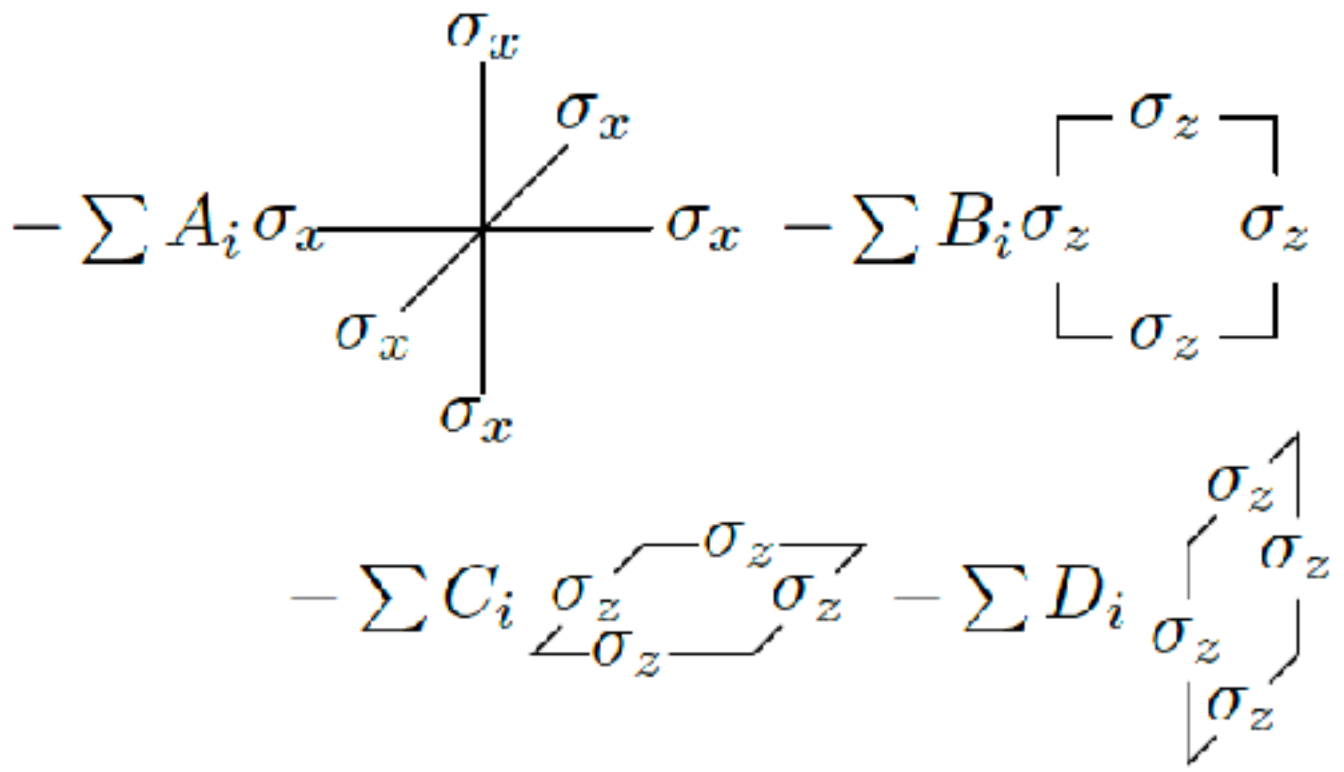
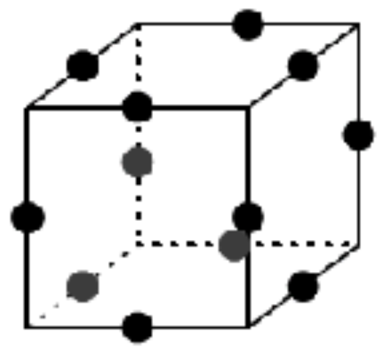
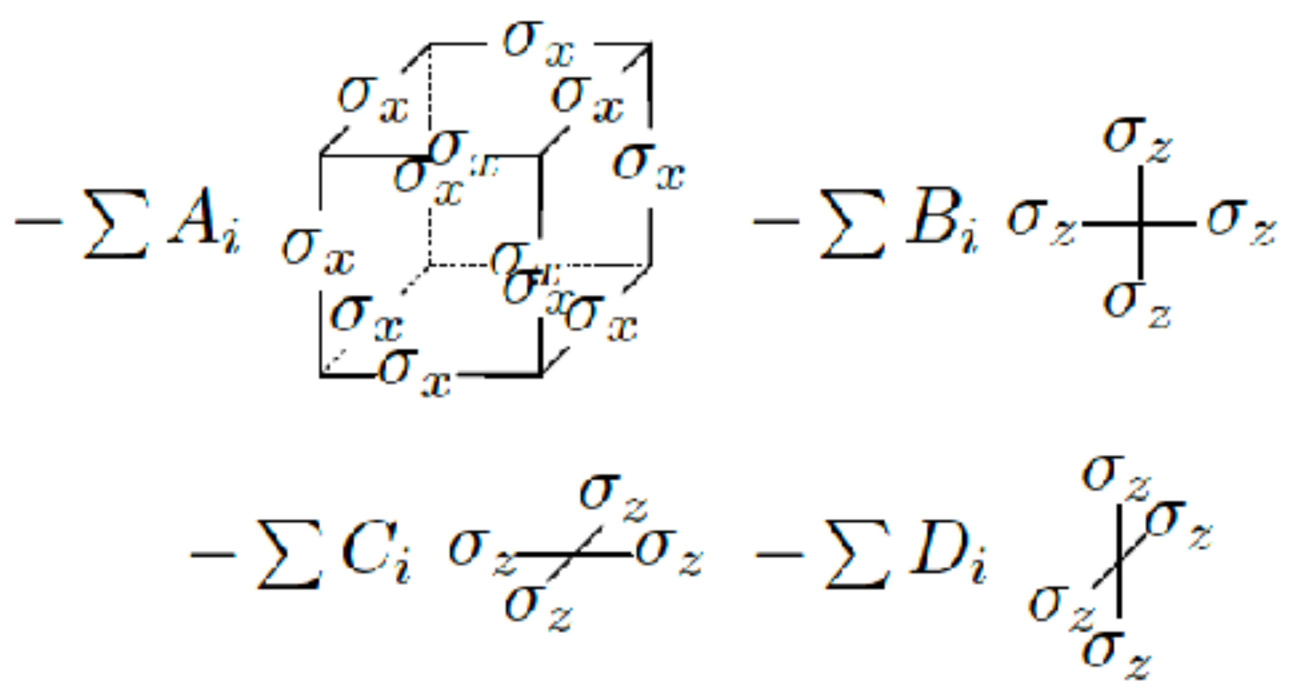
DUALITY OF OTHER CSS CODES

Original model	Lattice	Hamiltonian	Dual model
2D toric code			Two decoupled Ising chains
Rotated surface code			Non-interacting, single-spin
2D color code on a honeycomb lattice			

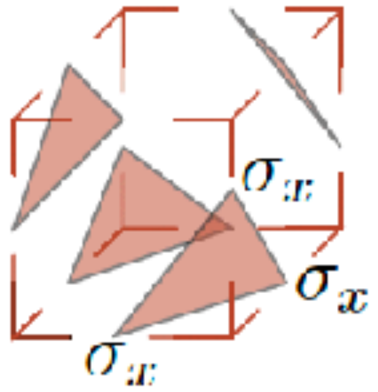
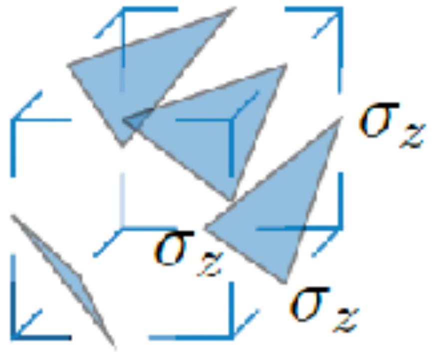
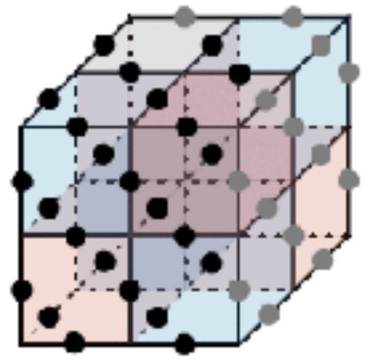
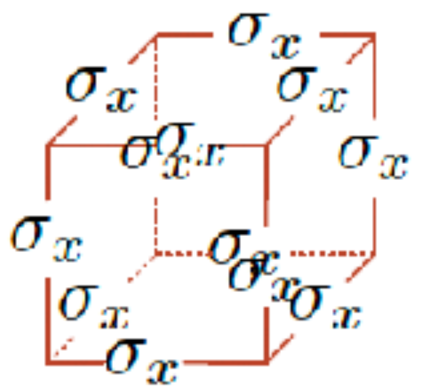
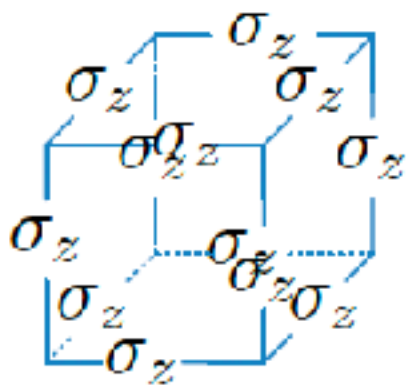


DUALITY OF OTHER CSS CODES

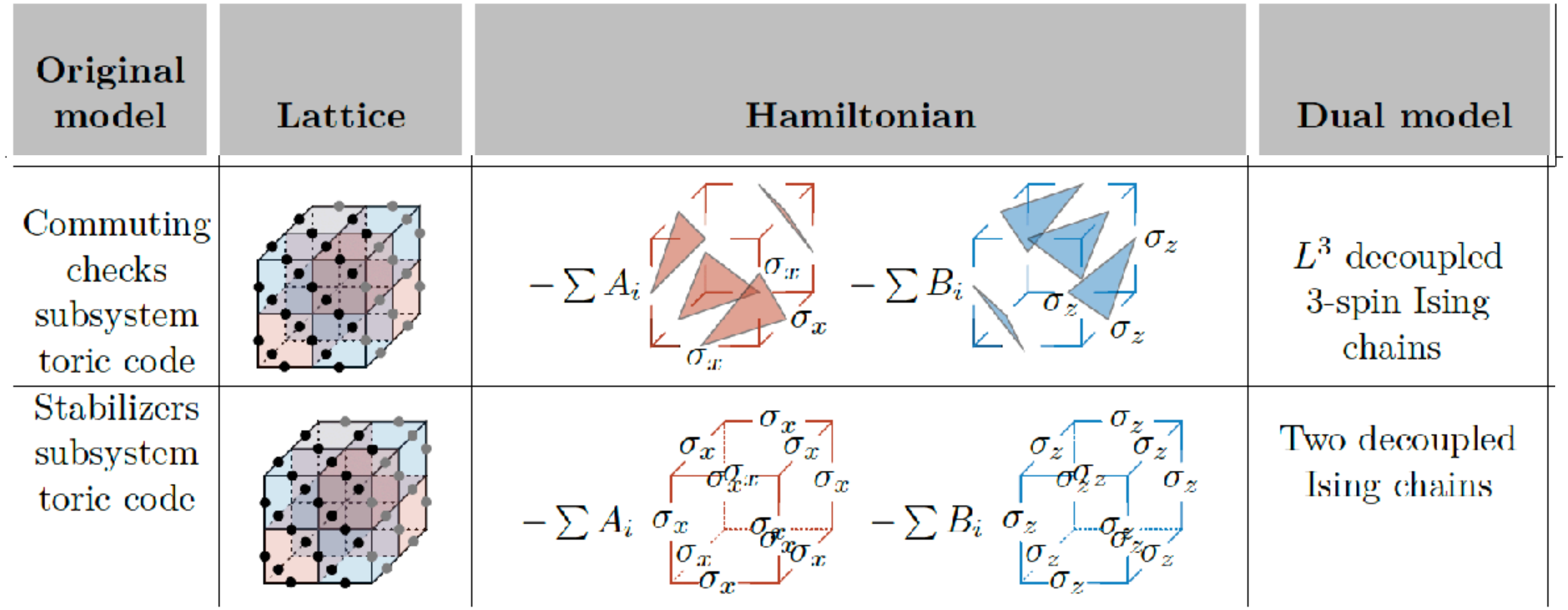


Original model	Lattice	Hamiltonian	Dual model	
Haah's Code			Two decoupled Ising chains	Periodic boundary conditions
3D toric code			Ising chain decoupled from a classical local model with constant degree interaction graph	Periodic boundary conditions
X-cube			L decoupled Ising chains and $L-1$ 1D decoupled nearest-neighbor systems	Cylindrical boundary conditions

DUALITY OF OTHER CSS CODES

Original model	**Lattice**	**Hamiltonian**	**Dual model**	
Commuting checks subsystem toric code		$-\sum A_i$  $-\sum B_i$ 	L^3 decoupled 3-spin Ising chains	Periodic boundary conditions
Stabilizers subsystem toric code		$-\sum A_i$  $-\sum B_i$ 	Two decoupled Ising chains	Periodic boundary conditions

DUALITY OF OTHER CSS CODES



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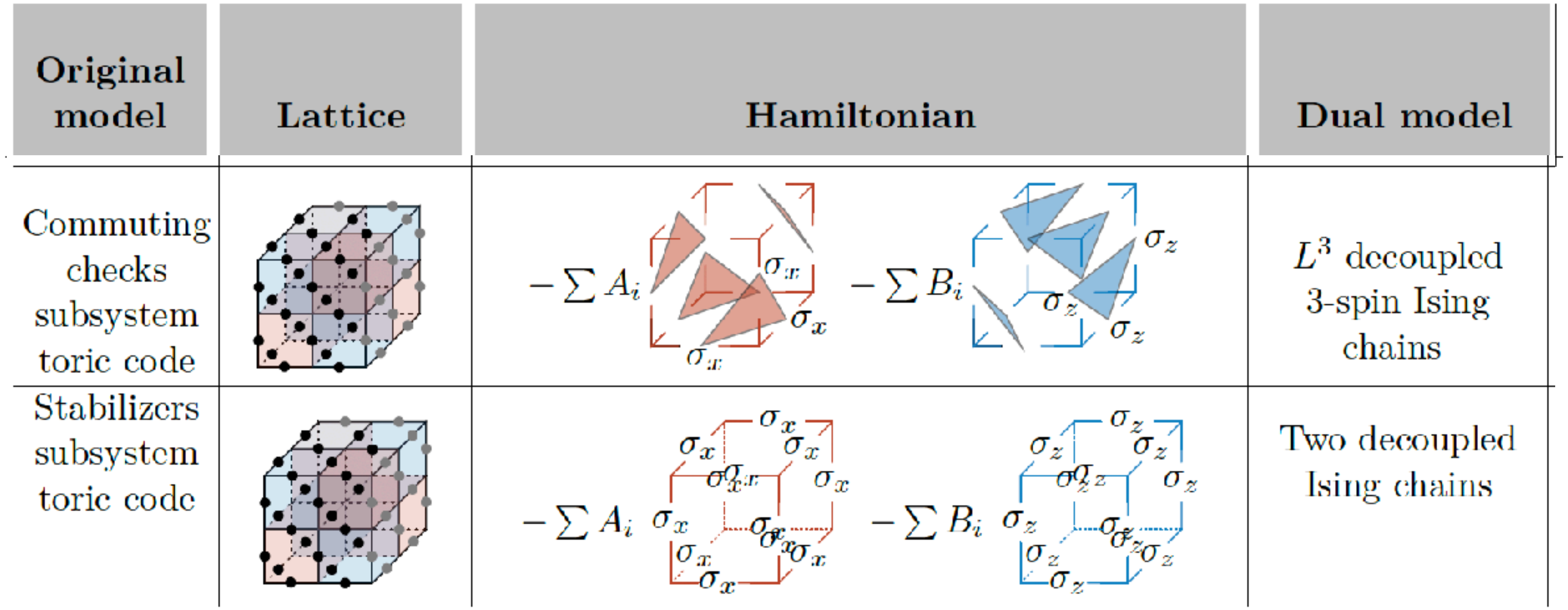
Periodic boundary conditions

Periodic boundary conditions

This is proven algorithmically for system sizes of order up to 10^5 qubits and conjectured in general.

DUALITY OF OTHER CSS CODES



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Consequence: All these models can be efficiently sampled for any $0 < \beta \leq \infty$, except for the 3D toric code, for which we only have efficient sampling at $0 < \beta \leq \beta_*$.

CONCLUSIONS

- The Gibbs state of the **2D toric code** is efficiently prepared at every positive temperature.

VIA DISSIPATION

Circuit depth $\mathcal{O}(|\Lambda|\text{polylog}|\Lambda|, \exp(\beta))$

Circuit complexity $\mathcal{O}(|\Lambda|^2\text{polylog}|\Lambda|, \exp(\beta))$

- Other consequences, such as **rapid loss of information**.
- Applicable to other models.
- Sets the basis to possible extensions to **other Lindbladians** and **non-commutative Hamiltonians**

VIA DUALITIES

Circuit complexity $\mathcal{O}(|\Lambda|^{3/2})$

- Very simple method and proof.
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THANKS FOR YOUR ATTENTION!